

MINKOWSKI'S THEOREM ON INDEPENDENT CONJUGATE UNITS

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ABSTRACT. We call a unit β in a finite, Galois extension l/\mathbb{Q} a Minkowski unit if the subgroup generated by β and its conjugates over \mathbb{Q} has maximum rank in the unit group of l . Minkowski showed the existence of such units in every Galois extension. We give a new proof of Minkowski's theorem and show that there exists a Minkowski unit $\beta \in l$ such that the Weil height of β is comparable with the sum of the heights of a fundamental system of units for l . Our proof implies a bound on the index of the subgroup generated by the algebraic conjugates of β in the unit group of l .

If k is an intermediate field such that

$$\mathbb{Q} \subseteq k \subseteq l,$$

and l/\mathbb{Q} and k/\mathbb{Q} are Galois extensions, we prove an analogous bound for the subgroup of relative units. In order to establish our results for relative units, a number of new ideas are combined with techniques from the geometry of numbers and the Galois action on places.

1. INTRODUCTION

Let l be an algebraic number field, O_l the ring of algebraic integers in l , and O_l^\times the multiplicative group of units in O_l . We write $\text{Tor}(O_l^\times)$ for the torsion subgroup of O_l^\times , which is the finite group of roots of unity in O_l^\times . Dirichlet's unit theorem asserts that there exists a nonnegative integer $r = r(l)$, called the *rank* of O_l^\times , such that

$$O_l^\times \cong \text{Tor}(O_l^\times) \oplus \mathbb{Z}^r.$$

Alternatively, there exists a finite collection $\eta_1, \eta_2, \dots, \eta_r$ of units such that each unit α in O_l^\times has a unique representation as

$$\alpha = \zeta \eta_1^{m_1} \eta_2^{m_2} \dots \eta_r^{m_r},$$

where ζ is an element of the torsion subgroup and m_1, m_2, \dots, m_r are integers. We say that $\eta_1, \eta_2, \dots, \eta_r$ form a *fundamental system* of units for O_l^\times . In this paper we assume that l is not \mathbb{Q} , and that l is not an imaginary, quadratic extension of \mathbb{Q} . This hypothesis insures that the rank $r = r(l)$ is positive.

If l/\mathbb{Q} is a Galois extension, we write $\text{Aut}(l/\mathbb{Q})$ for the Galois group of all automorphisms of l that fix \mathbb{Q} . And if β is a unit in O_l^\times , we write

$$(1.1) \quad \langle \sigma(\beta) : \sigma \in \text{Aut}(l/\mathbb{Q}) \rangle$$

for the subgroup of O_l^\times generated by the collection of all Galois conjugates of β . Minkowski [6] (see also [7, Theorem 3.26]) showed that it is always possible to select β so that the subgroup (1.1) has rank r , which is obviously the maximum possible

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rank of a subgroup. We call a unit β in a Galois extension a *Minkowski unit* if the subgroup (1.1) generated by its conjugates over \mathbb{Q} has maximum rank. Thus a unit β in a Galois extension l/\mathbb{Q} is a Minkowski unit if and only if there exists a subset of its conjugates

$$\{\sigma(\beta) : \sigma \in \text{Aut}(l/\mathbb{Q})\}$$

that has cardinality r and is multiplicatively independent. One of our objectives is to show that there exists a Minkowski unit β such that the Weil height of β is comparable with the sum of the heights of a fundamental system of units. If k is an intermediate field such that

$$\mathbb{Q} \subseteq k \subseteq l,$$

and k/\mathbb{Q} is a Galois extension, we prove an analogous bound for the subgroup of relative units.

Because roots of unity play no significant role in the results we establish, it will be convenient to work with an equivalent concept of Minkowski unit in the torsion free abelian group

$$(1.2) \quad F_l = O_l^\times / \text{Tor}(O_l^\times).$$

It follows that O_l^\times and F_l are both finitely generated abelian groups of the same rank $r = r(l)$, but F_l is free abelian. If

$$\alpha \mapsto \alpha \text{Tor}(O_l^\times)$$

is the canonical homomorphism, then the results we prove about subsets of F_l immediately imply corresponding results for subsets of O_l^\times by considering inverse images. If l/\mathbb{Q} is a Galois extension, then $\text{Aut}(l/\mathbb{Q})$ acts on O_l^\times , and also acts on the subgroup $\text{Tor}(O_l^\times)$. It follows that each automorphism σ in $\text{Aut}(l/\mathbb{Q})$ induces a well defined automorphism on the quotient group F_l . That is, if σ is an automorphism in $\text{Aut}(l/\mathbb{Q})$ and α is an element of O_l^\times , then

$$(\sigma, \alpha) \mapsto \sigma(\alpha)$$

defines an action of $\text{Aut}(l/\mathbb{Q})$ on O_l^\times , and

$$(\sigma, \alpha \text{Tor}(O_l^\times)) \mapsto \sigma(\alpha) \text{Tor}(O_l^\times)$$

defines a corresponding action of $\text{Aut}(l/\mathbb{Q})$ on cosets in F_l . Then it is obvious that the image of the subgroup (1.1) in F_l , is the subgroup

$$(1.3) \quad \langle \sigma(\beta) \text{Tor}(O_l^\times) : \sigma \in \text{Aut}(l/\mathbb{Q}) \rangle.$$

Moreover, if $\sigma_1, \sigma_2, \dots, \sigma_r$ are distinct automorphisms in $\text{Aut}(l/\mathbb{Q})$, then the subset

$$\{\sigma_1(\beta), \sigma_2(\beta), \dots, \sigma_r(\beta)\}$$

is multiplicatively independent in O_l^\times if and only if the subset

$$\{\sigma_1(\beta) \text{Tor}(O_l^\times), \sigma_2(\beta) \text{Tor}(O_l^\times), \dots, \sigma_r(\beta) \text{Tor}(O_l^\times)\}$$

is multiplicatively independent in F_l . Thus we may speak of a *Minkowski unit* in F_l , by which we understand a coset $\beta \text{Tor}(O_l^\times)$ in F_l such that the subgroup (1.3) has rank $r = r(l)$. However to simplify notation, in the remainder of the paper we usually write elements of F_l as coset representatives rather than as cosets.

We write l^\times for the multiplicative group of nonzero elements in l , and

$$h : l^\times \rightarrow [0, \infty)$$

for the absolute, logarithmic Weil height. This height is defined later in (4.2). As is well known, if α and ζ belong to l^\times and ζ is a root of unity, then we have $h(\alpha\zeta) = h(\alpha)$. It follows that h is constant on cosets of the quotient group

$$\mathcal{G}_l = l^\times / \text{Tor}(l^\times) = l^\times / \text{Tor}(O_l^\times),$$

and therefore h is well defined as a map

$$(1.4) \quad h : \mathcal{G}_l \rightarrow [0, \infty).$$

Elementary properties of the Weil height imply that the map

$$(\alpha, \beta) \mapsto h(\alpha\beta^{-1})$$

defines a metric on \mathcal{G}_l . In this work we will only have occasion to use the height h on the subgroup $F_l \subseteq \mathcal{G}_l$. Further properties of the Weil height on groups are discussed in [1], [3], and [9].

It follows from Minkowski's work in [6] that if F_l has positive rank $r = r(l)$, then there exists a coset representative β in F_l such that the multiplicative subgroup

$$(1.5) \quad \mathfrak{B} = \langle \sigma(\beta) : \sigma \in \text{Aut}(l/\mathbb{Q}) \rangle \subseteq F_l$$

generated by the orbit of β also has rank r . That is, β is a Minkowski unit in F_l , and therefore the index $[F_l : \mathfrak{B}]$ is finite. Here we give a proof of Minkowski's theorem which includes a bound on the index $[F_l : \mathfrak{B}]$, and a bound on the absolute logarithmic Weil height $h(\beta)$ of the Minkowski unit β .

Theorem 1.1. *Let l/\mathbb{Q} be a Galois extension, and assume that l has positive unit rank $r = r(l)$. Let $\eta_1, \eta_2, \dots, \eta_r$ be multiplicatively independent units in F_l , and write*

$$\mathfrak{A} = \langle \eta_1, \eta_2, \dots, \eta_r \rangle \subseteq F_l$$

for the subgroup of rank r that they generate. Then there exists a Minkowski unit β contained in \mathfrak{A} such that

$$(1.6) \quad h(\beta) \leq 2 \sum_{j=1}^r h(\eta_j).$$

Moreover, the subgroup (1.5) has index bounded by

$$(1.7) \quad \text{Reg}(l)[F_l : \mathfrak{B}] \leq ([l : \mathbb{Q}]h(\beta))^r,$$

where $\text{Reg}(l)$ is the regulator of l .

Theorem 1.1 is a simplified version of our more elaborate Theorem 5.1. In section 5 we define *special Minkowski units* in F_l with respect to a given archimedean place \widehat{w} of the field l . Lemma 5.2 shows that a special Minkowski unit is a Minkowski unit. Then Theorem 5.1 establishes the existence of a special Minkowski unit β with respect to a given archimedean place \widehat{w} , that satisfies the conclusion of Theorem 1.1, and has additional properties. For example, Theorem 5.1 identifies explicit subsets of the orbit of β that are multiplicatively independent. This is straightforward if l/\mathbb{Q} is a totally real Galois extension, but more complicated if l/\mathbb{Q} is a totally complex Galois extension.

If k is an intermediate field, that is, if

$$\mathbb{Q} \subseteq k \subseteq l,$$

then the norm from l^\times into k^\times induces a homomorphism

$$(1.8) \quad \text{norm}_{l/k} : F_l \rightarrow F_k.$$

The kernel of the homomorphism (1.8) is the subgroup

$$E_{l/k} = \{\alpha \in F_l : \text{norm}_{l/k}(\alpha) = 1\}$$

of relative units in F_l . (See [1], [4], and [5] for further properties of this subgroup.) Because F_k is a free abelian group, the kernel $E_{l/k}$ of the homomorphism (1.8) is a direct sum in F_l . And it follows from the discussion in section 6 that

$$(1.9) \quad r(l/k) = \text{rank } E_{l/k} = r(l) - r(k).$$

Therefore $E_{l/k}$ is a proper subgroup of F_l if and only if $1 \leq r(k) < r(l)$.

In general the Galois group $\text{Aut}(l/\mathbb{Q})$ does not act on the subgroup $E_{l/k}$, and therefore the simplest analogue of Minkowski's theorem cannot hold in $E_{l/k}$. If we assume that both l/\mathbb{Q} and k/\mathbb{Q} are Galois extensions, or equivalently, if we assume that $\text{Aut}(l/k)$ is a normal subgroup of $\text{Aut}(l/\mathbb{Q})$, then we show in Lemma 7.2 that the group $\text{Aut}(l/\mathbb{Q})$ does act on the subgroup $E_{l/k}$ of relative units. If both l/\mathbb{Q} and k/\mathbb{Q} are Galois extensions, we say that an element γ in $E_{l/k}$ is a *relative Minkowski unit* for the subgroup $E_{l/k}$, if the subgroup

$$\mathfrak{C} = \langle \sigma(\gamma) : \sigma \in \text{Aut}(l/\mathbb{Q}) \rangle \subseteq E_{l/k}$$

generated by the orbit of γ has rank equal to $r(l/k)$. Obviously (1.9) implies that $r(l/k)$ is the maximum possible rank of a subgroup in $E_{l/k}$. Our second main result establishes the existence of a relative Minkowski unit in $E_{l/k}$, and includes a bound on the height that is analogous to (1.6).

Theorem 1.2. *Let l/\mathbb{Q} and k/\mathbb{Q} be finite, Galois extensions such that*

$$\mathbb{Q} \subseteq k \subseteq l, \quad \text{and} \quad 1 \leq r(k) < r(l),$$

where $r(k)$ is the rank of F_k , and $r(l)$ is the rank of F_l . Let $\eta_1, \eta_2, \dots, \eta_{r(l)}$ be a basis for the group F_l .

- (i) *If l/\mathbb{Q} is a totally real Galois extension, then there exists a relative Minkowski unit γ in $E_{l/k}$ such that*

$$(1.10) \quad h(\gamma) \leq 4([l : k] - 1) \sum_{j=1}^{r(l)} h(\eta_j).$$

- (ii) *If l/\mathbb{Q} is a totally complex Galois extension, then there exists a relative Minkowski unit γ in $E_{l/k}$ such that*

$$(1.11) \quad h(\gamma) \leq 8([l : k] - 1) \sum_{j=1}^{r(l)} h(\eta_j).$$

We recall that a Galois extension l/\mathbb{Q} is either totally real or totally complex, and therefore (i) and (ii) in the statement of Theorem 1.2 cover all cases.

Theorem 1.2 is a simplified version of Theorem 9.1, which provides more precise information about the construction of the relative unit γ . A brief outline of the proof of Theorem 9.1, in case l/\mathbb{Q} is a totally real Galois extension, is as follows. We begin with a special Minkowski unit β contained in the group F_l , which exists by Theorem 5.1, and satisfies the inequality (1.6). In section 8 we define a bi-homomorphism

$$\Delta : F_l \times \mathbb{Z}^N \rightarrow F_l,$$

where N is the the number of archimedean places of l . Corollary 8.1 asserts that if β is a special Minkowski unit, and

$$\{f_1, f_2, \dots, f_R\}, \quad \text{where } R = r(l/k),$$

is a collection of linearly independent elements in a certain subgroup $\mathcal{L}_{l/k} \subseteq \mathbb{Z}^N$, then the units in the set

$$(1.12) \quad \{\Delta(\beta, f_1), \Delta(\beta, f_2), \dots, \Delta(\beta, f_R)\}$$

are multiplicatively independent relative units in $E_{l/k}$. Then we appeal to results in section 3 that establish the existence of an element λ in $\mathcal{L}_{l/k}$, and a subset

$$\{\psi_1, \psi_2, \dots, \psi_R\} \subseteq \text{Aut}(l/\mathbb{Q}),$$

such that the collection of conjugates

$$\{\psi_1(\Delta(\beta, \lambda)), \psi_2(\Delta(\beta, \lambda)), \dots, \psi_R(\Delta(\beta, \lambda))\}$$

has exactly the shape of the subset (1.12). It follows that

$$\gamma = \Delta(\beta, \lambda)$$

is a relative Minkowski unit for the subgroup $E_{l/k}$. The bound (1.10) follows from (1.6), the definition of λ in $\mathcal{L}_{l/k}$, and the definition of the bi-homomorphism Δ .

If l/\mathbb{Q} is a totally complex Galois extension the argument is similar. In this case we use a special Minkowski unit of the form $\beta\rho(\beta)$, where ρ is a certain element of $\text{Aut}(l/\mathbb{Q})$ of order 2. Then we show that

$$\gamma = \Delta(\beta\rho(\beta), \lambda).$$

is a relative Minkowski unit for the subgroup $E_{l/k}$. The bound (1.11) on the height of γ follows as in the previous case.

2. PRELIMINARY LEMMAS

In this section we prove four elementary lemmas about real matrices.

Lemma 2.1. *Let $A = (a_{mn})$ be a real, nonsingular, $N \times N$ matrix. Then there exists a point*

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix}$$

in \mathbb{Z}^N such that

$$(2.1) \quad 0 < \sum_{n=1}^N a_{mn} \xi_n \leq \sum_{n=1}^N |a_{mn}| \quad \text{for each } m = 1, 2, \dots, N.$$

Proof. For each positive integer l , define the (column) vector $\mathbf{u}^{(l)}$ in \mathbb{R}^N by

$$(2.2) \quad u_m^{(l)} = \left(\frac{1}{2} + \frac{1}{l}\right) \sum_{n=1}^N |a_{mn}| \quad \text{for each } m = 1, 2, \dots, N.$$

Then set $\mathbf{v}^{(l)} = A^{-1} \mathbf{u}^{(l)}$, and select $\boldsymbol{\xi}^{(l)}$ in \mathbb{Z}^N so that

$$-\frac{1}{2} \leq v_n^{(l)} - \xi_n^{(l)} \leq \frac{1}{2} \quad \text{for each } n = 1, 2, \dots, N.$$

It follows that

$$(2.3) \quad \left| \sum_{n=1}^N a_{mn} \xi_n^{(l)} - u_m^{(l)} \right| = \left| \sum_{n=1}^N a_{mn} (\xi_n^{(l)} - v_n^{(l)}) \right| \leq \frac{1}{2} \sum_{n=1}^N |a_{mn}|$$

for each $m = 1, 2, \dots, N$. Combining (2.2) and (2.3) we get

$$-\frac{1}{2} \sum_{n=1}^N |a_{mn}| \leq \sum_{n=1}^N a_{mn} \xi_n^{(l)} - \left(\frac{1}{2} + \frac{1}{l}\right) \sum_{n=1}^N |a_{mn}| \leq \frac{1}{2} \sum_{n=1}^N |a_{mn}|,$$

and therefore

$$(2.4) \quad \frac{1}{l} \sum_{n=1}^N |a_{mn}| \leq \sum_{n=1}^N a_{mn} \xi_n^{(l)} \leq \left(1 + \frac{1}{l}\right) \sum_{n=1}^N |a_{mn}|$$

for each $m = 1, 2, \dots, N$.

Now observe that the set

$$\left\{ \mathbf{x} \in \mathbb{R}^N : 0 < \sum_{n=1}^N a_{mn} x_n \leq 2 \sum_{n=1}^N |a_{mn}| \text{ for } m = 1, 2, \dots, N \right\}$$

is bounded, and therefore it intersects \mathbb{Z}^N in only finitely many points. It follows from (2.4) that the map $l \mapsto \xi^{(l)}$ is constant for l in an infinite subset of $\{1, 2, 3, \dots\}$. Letting ξ in \mathbb{Z}^N denote such a constant, we conclude that

$$0 < \sum_{n=1}^N a_{mn} \xi_n \leq \sum_{n=1}^N |a_{mn}| \text{ for each } m = 1, 2, \dots, N,$$

and the lemma plainly follows. \square

The following is a variant of a lemma due to Minkowski.

Lemma 2.2. *Let $A = (a_{mn})$ be a real, $N \times N$ matrix such that*

$$(2.5) \quad 0 < \sum_{m=1}^N a_{mn}, \quad \text{for each } n = 1, 2, \dots, N,$$

and

$$a_{mn} < 0, \quad \text{for } m \neq n.$$

Then A is nonsingular.

Proof. Assume that $\det A = 0$. Then there exists a point $\mathbf{x} \neq \mathbf{0}$ in \mathbb{R}^N such that

$$(2.6) \quad 0 = \sum_{m=1}^N a_{mn} x_m, \quad \text{for each } n = 1, 2, \dots, N.$$

By replacing \mathbf{x} with $-\mathbf{x}$ if necessary, we can select an integer r such that

$$(2.7) \quad 0 < x_r = \max\{|x_n| : n = 1, 2, \dots, N\}.$$

Then applying (2.6) with $n = r$, and using (2.7), we find that

$$\begin{aligned} 0 &= \sum_{m=1}^N a_{mr} x_m = a_{rr} x_r + \sum_{\substack{m=1 \\ m \neq r}}^N a_{mr} x_m \\ &\geq a_{rr} x_r + \sum_{\substack{m=1 \\ m \neq r}}^N a_{mr} x_r = x_r \sum_{m=1}^N a_{mr} > 0. \end{aligned}$$

We conclude that $\mathbf{x} \neq \mathbf{0}$ does not exist, and therefore A is nonsingular. \square

If A is a real, $N \times N$ matrix then the \mathbb{Q} -rank of A is the number of \mathbb{Q} -linearly independent rows (or columns) of A . Similarly, the \mathbb{R} -rank of A is the number of \mathbb{R} -linearly independent rows (or columns) of A . In general the \mathbb{Q} -rank of A is greater than or equal to the \mathbb{R} -rank of A . Of course $\det A \neq 0$ if and only if the \mathbb{R} -rank of A is N , and this implies that the \mathbb{Q} -rank is also N . But the matrix

$$A = \begin{pmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{pmatrix}$$

has \mathbb{Q} -rank equal to 2, and \mathbb{R} -rank equal to 1.

Lemma 2.3. *Let $A = (a_{mn})$ be a real, $N \times N$ matrix such that*

$$(2.8) \quad \sum_{m=1}^N a_{mn} = 0, \quad \text{for each } n = 1, 2, \dots, N,$$

and

$$(2.9) \quad \sum_{n=1}^N a_{mn} = 0, \quad \text{for each } m = 1, 2, \dots, N.$$

Let $A_{(m,n)}$ denote the $(N-1) \times (N-1)$ submatrix of A obtained by removing the row indexed by m , and removing the column indexed by n . Then there exists a real constant c such that

$$(2.10) \quad (-1)^{m+n} \det A_{(m,n)} = c$$

for each pair (m,n) . Moreover, we have $c \neq 0$ if and only if the \mathbb{R} -rank of A and the \mathbb{Q} -rank of A are both equal to $N-1$.

Proof. It is clear from (2.8) that the \mathbb{Q} -rank of A is less than or equal to $N-1$, and $\det A = 0$. If the \mathbb{R} -rank is less than or equal to $N-2$ then (2.10) holds with $c = 0$. Thus we assume for the remainder of the proof that the \mathbb{R} -rank of A , and the \mathbb{Q} -rank of A , are both equal to $N-1$. Then it follows from (2.8) that

$$(2.11) \quad \sum_{m=1}^N a_{mn} x_m = 0, \quad \text{for each } n = 1, 2, \dots, N,$$

if and only if $m \mapsto x_m$ is constant, and it follows from (2.9) that

$$(2.12) \quad \sum_{n=1}^N a_{mn} y_n = 0, \quad \text{for each } m = 1, 2, \dots, N,$$

if and only if $n \mapsto y_n$ is constant. We also have

$$(-1)^{i+j} \det A_{(i,j)} \neq 0$$

for some pair of indices (i, j) .

Recall that the Laplace expansion of the determinant along columns is

$$(2.13) \quad \sum_{m=1}^N (-1)^{m+t} a_{ms} \det A_{(m,t)} = \begin{cases} \det A & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

As $\det A = 0$, it follows from (2.11) and (2.13) that for each index t the function

$$m \mapsto (-1)^{m+t} \det A_{(m,t)} = c(t)$$

is a real constant that depends only on t . Similarly, the expansion along rows is

$$(2.14) \quad \sum_{n=1}^N (-1)^{v+n} a_{un} \det A_{(v,n)} = \begin{cases} \det A & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

Again we have $\det A = 0$, and therefore (2.12) and (2.14) imply that for each index v the function

$$n \mapsto (-1)^{v+n} \det A_{(v,n)} = d(v)$$

is a real constant that depends only on v . We have shown that

$$(2.15) \quad c(n) = (-1)^{m+n} \det A_{(m,n)} = d(m)$$

for each pair of indices (m, n) . It follows from (2.15) that both

$$m \mapsto d(m), \quad \text{and} \quad n \mapsto c(n),$$

are constant. Taking $i = m$ and $j = n$ shows that this constant is

$$(-1)^{i+j} \det A_{(i,j)} \neq 0.$$

This proves the lemma. □

Lemma 2.2 and Lemma 2.3 can be combined to establish the following general result.

Lemma 2.4. *Let $A = (a_{mn})$ be a real, $N \times N$ matrix such that*

$$(2.16) \quad 0 = \sum_{m=1}^N a_{mn}, \quad \text{for each } n = 1, 2, \dots, N,$$

and

$$(2.17) \quad a_{mn} < 0, \quad \text{for } m \neq n.$$

Then A satisfies the following conditions.

- (i) *The \mathbb{Q} -rank of A , and the \mathbb{R} -rank of A , are both equal to $N - 1$.*
- (ii) *If $\mathbf{y} \neq \mathbf{0}$ is a point in \mathbb{R}^N such that*

$$(2.18) \quad 0 = \sum_{n=1}^N a_{mn} y_n, \quad \text{for each } m = 1, 2, \dots, N,$$

then the co-ordinates y_1, y_2, \dots, y_N are all positive, or all negative.

- (iii) *For each pair (m, n) the submatrix $A_{(m,n)}$ is nonsingular.*

Proof. It is clear from (2.16) that the rows of A are \mathbb{Q} -linearly dependent, and therefore the \mathbb{Q} -rank of A is at most $N - 1$. Thus it suffices to show that the \mathbb{R} -rank of A is at least $N - 1$. Let \mathbf{u} in \mathbb{Z}^N be such that $u_n = 1$ for each $n = 1, 2, \dots, N$, and let \mathfrak{N} be the null space

$$\mathfrak{N} = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x}^T A = \mathbf{0}^T\}.$$

Then \mathfrak{N} is an \mathbb{R} -linear subspace of \mathbb{R}^N , and by (2.16) the subspace \mathfrak{N} contains \mathbf{u} . Assume that $\mathbf{w} \neq \mathbf{0}$ is also in \mathfrak{N} . Let

$$w_i = \min\{w_m : m = 1, 2, \dots, N\}, \quad \text{and} \quad w_j = \max\{w_m : m = 1, 2, \dots, N\},$$

and assume that $w_i < w_j$. Then $\mathbf{w} - w_i \mathbf{u}$ belongs to \mathfrak{N} , and therefore

$$(2.19) \quad 0 = \sum_{\substack{m=1 \\ m \neq i}}^N (w_m - w_i) a_{mn}, \quad \text{for each } n = 1, 2, \dots, N.$$

Taking $n = i$ in (2.19), we find that

$$0 = \sum_{\substack{m=1 \\ m \neq i}}^N (w_m - w_i) a_{mi} \leq (w_j - w_i) a_{ji} < 0,$$

which is impossible. Therefore $w_i = w_j$, and \mathbf{w} is a real multiple of \mathbf{u} . Hence \mathfrak{N} is an \mathbb{R} -linear subspace of dimension 1. It follows that the \mathbb{R} -rank of A is $N - 1$.

Let $1 \leq r \leq N$ and let $A_{(r,r)}$ be the $(N - 1) \times (N - 1)$ submatrix of A obtained by removing the row indexed by r , and the column indexed by r . It follows from (2.16) and (2.17) that $A_{(r,r)}$ satisfies the hypotheses of Lemma 2.2, but with N replaced by $N - 1$. We conclude from Lemma 2.2 that each submatrix $A_{(r,r)}$ is nonsingular. Now suppose that $\mathbf{y} \neq \mathbf{0}$ satisfies (2.18), and $y_r = 0$. Let \mathbf{z} be the (column) vector in \mathbb{R}^{N-1} obtained from \mathbf{y} by removing $y_r = 0$. Then we have

$$\mathbf{z} \neq \mathbf{0}, \quad \text{and} \quad A_{(r,r)} \mathbf{z} = \mathbf{0},$$

which contradicts the fact that the submatrix $A_{(r,r)}$ is nonsingular. We have shown that if $\mathbf{y} \neq \mathbf{0}$ satisfies (2.18), then $y_r \neq 0$ for each $r = 1, 2, \dots, N$. Next we define

$$I = \{n : 0 < y_n\}, \quad \text{and} \quad J = \{n : y_n < 0\},$$

and we assume that both I and J are not empty. It follows from (2.16) and (2.17) that

$$(2.20) \quad 0 < \sum_{m \in I} a_{mn}, \quad \text{if } n \text{ belongs to } I,$$

and

$$(2.21) \quad \sum_{m \in I} a_{mn} < 0, \quad \text{if } n \text{ belongs to } J.$$

Applying (2.18), (2.20), and (2.21), we find that

$$\begin{aligned}
0 &= \sum_{m \in I} \sum_{n=1}^N a_{mn} y_n \\
&= \sum_{n \in I} \left(\sum_{m \in I} a_{mn} \right) y_n + \sum_{n \in J} \left(\sum_{m \in I} a_{mn} \right) y_n \\
&> 0.
\end{aligned}$$

The contradiction implies that either I or J is empty. That is, the co-ordinates y_1, y_2, \dots, y_N are all positive, or all negative.

We continue to suppose that $\mathbf{y} \neq \mathbf{0}$ is a point in \mathbb{R}^N that satisfies (2.18). Then we let $[y_n]$ denote the $N \times N$ diagonal matrix with y_1, y_2, \dots, y_N as consecutive diagonal entries. We have already verified (ii), and therefore

$$(2.22) \quad Y = \det[y_n] = y_1 y_2 \cdots y_N \neq 0.$$

Let B denote the $N \times N$ real matrix

$$(2.23) \quad B = A[y_n] = (a_{mn} y_n).$$

From (2.16) we get

$$0 = \sum_{m=1}^N a_{mn} y_n, \quad \text{for each } n = 1, 2, \dots, N,$$

and (2.18) asserts that

$$0 = \sum_{n=1}^N a_{mn} y_n, \quad \text{for each } m = 1, 2, \dots, N.$$

It follows that B satisfies the hypotheses of Lemma 2.3. As $[y_n]$ is nonsingular, B has rank $(N-1)$. Hence there exists a constant $b \neq 0$ such that

$$(-1)^{m+n} \det B_{(m,n)} = b$$

for each $(N-1) \times (N-1)$ submatrix $B_{(m,n)}$. Using (2.22) and (2.23), we find that

$$y_n \det B_{(m,n)} = Y \det A_{(m,n)}$$

for each integer pair (m, n) . This shows that each submatrix $A_{(m,n)}$ is nonsingular, and also establishes the identity

$$(-1)^{m+n} Y \det A_{(m,n)} = b y_n$$

for each integer pair (m, n) and a real constant $b \neq 0$. □

3. FUNCTIONS ON FINITE GROUPS

Throughout this section we assume that G is a finite group with subgroups $H \subseteq G$ and $K \subseteq G$. We assume that H is a normal subgroup of G , and we write $N = [G : K]$ for the index of K in G . We consider two cases: either $H \cap K = \{1\}$, or $K \subseteq H$. If $H \cap K = \{1\}$, then

$$HK = KH = \{hk : h \in H \text{ and } k \in K\},$$

and the map

$$(h, k) \mapsto hk$$

from $H \times K$ into HK is bijective. Because H is normal in G , the subset HK is a subgroup of G , (see [2, Chapter 2, Proposition (8.6)]). If $K \subseteq H$ the situation is simpler because $HK = H$. In both cases we define

$$I = [G : HK], \quad \text{and} \quad J = [HK : K],$$

so that $IJ = N$.

We write \mathbb{Z}^N for the free abelian group of rank N , and we identify elements of this group with functions

$$f : G \rightarrow \mathbb{Z}$$

that are constant on each left coset of K . As K has N left cosets in G , it is clear that this group of functions is free abelian of rank N . Alternatively, we write G/K for the collection of all left cosets of K in G , and we identify elements of \mathbb{Z}^N with functions

$$f : G/K \rightarrow \mathbb{Z}.$$

As is well known, the group G acts on the set G/K of all left cosets of K in G by multiplication on the left, (see [2, Chapter 5, section 6 and section 7]). This action induces an action of G on \mathbb{Z}^N as follows: if g belongs to G and $x \mapsto f(x)$ is a function in \mathbb{Z}^N , we write $[g, f]$ for the action of g on f , and we define this element of \mathbb{Z}^N by

$$(3.1) \quad x \mapsto [g, f](x) = f(g^{-1}x).$$

If 1 is the identity element in G then

$$(3.2) \quad x \mapsto [1, f](x) = f(x)$$

is obvious. And if g_1 and g_2 belong to G then

$$(3.3) \quad \begin{aligned} [g_1, [g_2, f]](x) &= [g_2, f](g_1^{-1}x) \\ &= f(g_2^{-1}g_1^{-1}x) \\ &= f((g_1g_2)^{-1}x) \\ &= [g_1g_2, f](x). \end{aligned}$$

It follows from (3.2) and (3.3) that (3.1) defines an action of the group G on the collection of functions \mathbb{Z}^N .

Let $\{s_1, s_2, \dots, s_I\}$ be a transversal for the left cosets of the subgroup HK in G , so that

$$G = \bigcup_{i=1}^I s_i HK$$

is a disjoint union. Let $\{t_1, t_2, \dots, t_J\}$ be a transversal for the left cosets of the subgroup K in HK , so that

$$(3.4) \quad HK = \bigcup_{j=1}^J t_j K$$

is a disjoint union. It follows that

$$G = \bigcup_{i=1}^I s_i HK = \bigcup_{i=1}^I \bigcup_{j=1}^J s_i t_j K,$$

and therefore

$$(3.5) \quad \{s_i t_j : i = 1, 2, \dots, I \text{ and } j = 1, 2, \dots, J\}$$

is a transversal for left cosets of the subgroup K in G . Thus a function f in \mathbb{Z}^N is uniquely determined by its values on the distinct left coset representatives (3.5). We define the subgroup

$$(3.6) \quad \mathcal{L} = \left\{ f \in \mathbb{Z}^N : \sum_{j=1}^J f(s_i t_j) = 0 \text{ for each } i = 1, 2, \dots, I \right\}.$$

Because f in \mathbb{Z}^N is constant on each left coset of K , the identity (3.4) implies that

$$(3.7) \quad \sum_{g \in s_i HK} f(g) = |K| \sum_{j=1}^J f(s_i t_j)$$

for each left coset $s_i HK$ in G . Thus f in \mathbb{Z}^N belongs to the subgroup \mathcal{L} if and only if the sum of the values that f takes on each coset of HK is zero. In particular, the choice of transversals $\{s_1, s_2, \dots, s_I\}$ and $\{t_1, t_2, \dots, t_J\}$ does not effect the definition of the subgroup \mathcal{L} .

As the subsets

$$\{s_i t_j : j = 1, 2, \dots, J\}, \quad \text{where } i = 1, 2, \dots, I,$$

are disjoint, the I linear equations satisfied by functions f in \mathcal{L} are clearly independent. Hence the subgroup \mathcal{L} has rank $N - I$. We note that

$$N - I = I(J - 1).$$

We have observed that the group G acts on the group \mathbb{Z}^N by (3.1). We now show that the subgroup \mathcal{L} is invariant under this action. That is, G acts on \mathcal{L} .

Lemma 3.1. *Let g be an element of G , and let f be a function in \mathcal{L} . Then the function*

$$x \mapsto [g, f](x) = f(g^{-1}x)$$

belongs to \mathcal{L} .

Proof. For each $i = 1, 2, \dots, I$ we have

$$(3.8) \quad |K| \sum_{j=1}^J [g, f](s_i t_j) = |K| \sum_{j=1}^J f(g^{-1} s_i t_j) = \sum_{h \in g^{-1} s_i HK} f(h).$$

But the sum on the right of (3.8) is zero because f belongs to \mathcal{L} and $g^{-1} s_i HK$ is a left coset of HK . \square

We now consider and solve the following problem: construct a function λ in \mathcal{L} such that the subgroup

$$(3.9) \quad \langle [g, \lambda] : g \in G \rangle$$

generated by the orbit of λ under the action of G , has rank $N - I$ in \mathcal{L} . We define λ by

$$(3.10) \quad \lambda(g) = \begin{cases} J - 1 & \text{if } g \text{ belongs to } K, \\ -1 & \text{if } g \text{ belongs to } HK, \text{ but does not belong to } K, \\ 0 & \text{if } g \text{ belongs to } G, \text{ but does not belong to } HK. \end{cases}$$

We will prove that for this choice of λ , the subgroup (3.9) has rank $N - I$.

If $s_i HK$ is a left coset of HK , but not equal to HK , then

$$(3.11) \quad \sum_{g \in s_i HK} \lambda(g) = 0$$

is obvious because each term in the sum is zero. When we sum over the subgroup HK we find that

$$(3.12) \quad \sum_{g \in HK} \lambda(g) = |K| \sum_{j=1}^J \lambda(t_j) = |K|(J - 1 - (J - 1)) = 0.$$

It follows that λ belongs to \mathcal{L} .

Lemma 3.2. *Let λ in \mathcal{L} be defined by (3.10), and let $\{t_1, t_2, \dots, t_J\}$ be a transversal for the left cosets of K in HK . Then each subset of cardinality $J - 1$, contained in the collection of functions*

$$\{[t_j, \lambda] : j = 1, 2, \dots, J\},$$

is linearly independent.

Proof. Let μ be the function λ with domain restricted to those left cosets of K that are contained in HK , so that

$$(3.13) \quad \mu(h) = \begin{cases} J - 1 & \text{if } h \text{ belongs to } K, \\ -1 & \text{if } h \text{ belongs to } HK, \text{ but does not belong to } K. \end{cases}$$

For x in HK we have

$$\mu(t_j^{-1}x) = \lambda(t_j^{-1}x) = [t_j, \lambda](x).$$

Therefore it suffices to show that each subset of cardinality $J - 1$, contained in the collection of functions

$$(3.14) \quad \{\mu(t_j^{-1}x) : j = 1, 2, \dots, J\},$$

is linearly independent. Here each function $x \mapsto \mu(t_j^{-1}x)$ is defined on the set of J distinct left cosets of K in HK , and these cosets are represented by the elements of the transversal $\{t_1, t_2, \dots, t_J\}$.

Let

$$M = (\mu(t_j^{-1}t_i))$$

be the $J \times J$ integer matrix, where $i = 1, 2, \dots, J$ indexes rows, and $j = 1, 2, \dots, J$ indexes columns. If $i \neq j$ then $t_j^{-1}t_i$ does not belong to K , and it follows from (3.13) that

$$\mu(t_j^{-1}t_i) = -1 < 0.$$

For each $j = 1, 2, \dots, J$ the elements in the set

$$\{t_j^{-1}t_1, t_j^{-1}t_2, \dots, t_j^{-1}t_J\}$$

form a transversal for the left cosets of K in HK . Hence we get

$$\sum_{i=1}^J \mu(t_j^{-1}t_i) = (J - 1) - \sum_{\substack{i=1 \\ i \neq j}}^J 1 = 0.$$

We have shown that M satisfies the hypotheses (2.16) and (2.17) in the statement of Lemma 2.4. Hence by that result the matrix M has \mathbb{Q} -rank and \mathbb{R} -rank equal to $J - 1$. If z_1, z_2, \dots, z_J are integers, not all of which are zero, such that

$$\sum_{j=1}^J z_j \mu(t_j^{-1} t_i) = 0, \quad \text{for each } i = 1, 2, \dots, J,$$

then (ii) of Lemma 2.4 asserts that the integers z_j are all positive or all negative. In particular, each subset of cardinality $J - 1$, contained in the collections of functions (3.14), is linearly independent. \square

We are now able to prove that the subgroup (3.9) has rank $N - I$. This is contained in the following more precise result.

Lemma 3.3. *Let λ in \mathcal{L} be defined by (3.10), let $\{s_1, s_2, \dots, s_I\}$ be a transversal for the left cosets of HK in G , and let $\{t_1, t_2, \dots, t_J\}$ be a transversal for the left cosets of K in HK . For each $i = 1, 2, \dots, I$, let*

$$\mathcal{J}_i \subseteq \{1, 2, \dots, J\}$$

be a subset of cardinality $|\mathcal{J}_i| = J - 1$. Then the collection of $N - I$ functions

$$(3.15) \quad \{[s_i t_j, \lambda] : i = 1, 2, \dots, I, \text{ and } j \in \mathcal{J}_i\} \subseteq \mathcal{L},$$

is linearly independent. Moreover, for each $i = 1, 2, \dots, I$, the $J - 1$ functions in the subcollection

$$\{[s_i t_j, \lambda] : j \in \mathcal{J}_i\},$$

are supported on the left coset $s_i HK$.

Proof. It follows from the definition (3.10) that the function λ is supported on the subgroup HK . Hence for each $i = 1, 2, \dots, I$, the function

$$x \mapsto [s_i t_j, \lambda](x) = \lambda((s_i t_j)^{-1} x)$$

is supported on the left coset

$$s_i t_j HK = s_i HK.$$

This verifies the last statement in the lemma.

Let

$$\{z(i, j) : i = 1, 2, \dots, I \text{ and } j \in \mathcal{J}_i\}$$

be a collection of integers, not all of which are zero. Assume that the function

$$x \mapsto \sum_{i=1}^I \sum_{j \in \mathcal{J}_i} z(i, j) \lambda((s_i t_j)^{-1} x)$$

is identical zero on G . For each $i = 1, 2, \dots, I$ the function

$$(3.16) \quad x \mapsto \sum_{j \in \mathcal{J}_i} z(i, j) \lambda((s_i t_j)^{-1} x)$$

is supported on the left coset $s_i HK$, and the left cosets

$$s_i HK, \quad \text{where } i = 1, 2, \dots, I,$$

are obviously disjoint. Hence each function (3.16) is identically zero on G .

Because $\{t_1, t_2, \dots, t_J\}$ is a transversal for the left cosets of K in HK , for $i = 1, 2, \dots, I$, each left coset of K in $s_i HK$ is represented by a unique element in the set

$$\{s_i t_1, s_i t_2, \dots, s_i t_J\}.$$

It follows that for each $i = 1, 2, \dots, I$, we have

$$(3.17) \quad 0 = \sum_{j \in \mathcal{J}_i} z(i, j) \lambda((s_i t_j)^{-1} s_i t_k) = \sum_{j \in \mathcal{J}_i} z(i, j) \lambda(t_j^{-1} t_k)$$

for each $k = 1, 2, \dots, J$. But \mathcal{J}_i has cardinality $J - 1$, and therefore the identity (3.17) contradicts the statement of Lemma 3.2. We conclude that the integers $z(i, j)$ do not exist. That is, the functions in the collection (3.15) are linearly independent.

As the functions in the collection (3.15) are linearly independent, they are distinct, and the cardinality of the collection (3.15) is

$$\sum_{i=1}^I |\mathcal{J}_i| = I(J - 1) = IJ - I = N - I.$$

This proves the lemma. \square

4. THE GALOIS ACTION ON PLACES

We assume that l and k are algebraic number fields such that

$$\mathbb{Q} \subseteq k \subseteq l.$$

At each place v of k we write k_v for the completion of k at v , so that k_v is a local field. We select two absolute values $\|\cdot\|_v$ and $|\cdot|_v$ from the place v . The absolute value $\|\cdot\|_v$ extends the usual archimedean or non-archimedean absolute value on the subfield \mathbb{Q} . Then $|\cdot|_v$ must be a power of $\|\cdot\|_v$, and we set

$$(4.1) \quad |\cdot|_v = \|\cdot\|_v^{d_v/d},$$

where $d_v = [k_v : \mathbb{Q}_v]$ is the local degree of the extension, and $d = [k : \mathbb{Q}]$ is the global degree. In a similar manner we write w for a place of l , l_w for the completion of l at w , and we normalize two absolute values $\|\cdot\|_w$ and $|\cdot|_w$ from the place w in a similar manner. We write $w|v$ when $\|\cdot\|_w$ extends the absolute value $\|\cdot\|_v$ from k to l . Then we write $W_v(l/k)$ for the finite set of all places w of l such that $w|v$.

With these normalizations the height of an algebraic number $\alpha \neq 0$ that belongs to l is given by

$$(4.2) \quad h(\alpha) = \sum_w \log^+ |\alpha|_w = \frac{1}{2} \sum_w |\log |\alpha|_w|.$$

Each sum in (4.2) is over the set of all places w of l , and the equality between the two sums follows from the product formula. Then $h(\alpha)$ depends on the algebraic number $\alpha \neq 0$, but it does not depend on the number field l that contains α . We have already noted in (1.4) that the height is well defined as a map

$$h : \mathcal{G}_l \rightarrow [0, \infty).$$

If l/k is a finite, Galois extension then the Galois group $\text{Aut}(l/k)$ acts transitively on the set $W_v(l/k)$ of places w of l that lie above a fixed place v of k (see Tate [8]). If σ is an element of $\text{Aut}(l/k)$ and w is a place of l , then σw is the unique place of l that satisfies the identity

$$(4.3) \quad \|\sigma^{-1}(\gamma)\|_w = \|\gamma\|_{\sigma w}$$

for each γ in l . Because σ fixes elements of k , we find that the restriction of (4.3) to k is equal to the restriction of $\|\cdot\|_w$ to k . That is, σw and w are both places in the set $W_v(l/k)$. For a Galois extension all local degrees over a fixed place of k are equal. Alternatively, the map $w \mapsto [l_w : k_v]$ is constant on places w that belong to $W_v(l/k)$. This observation easily implies that

$$(4.4) \quad |\sigma^{-1}(\gamma)|_w = |\gamma|_{\sigma w}$$

for each γ in l .

Now assume that l/\mathbb{Q} is a finite, Galois extension and k is an intermediate field. We write

$$G = \text{Aut}(l/\mathbb{Q}), \quad \text{and} \quad H = \text{Aut}(l/k),$$

so that $H \subseteq G$ is a subgroup of G , and H is the group of automorphisms attached to the Galois extension l/k . If w is a place of l , if v is a place of k such that $w|v$, if l_w is the completion of l at w , and k_v is the completion of k at v , then l_w/k_v is a Galois extension. It can be shown (see Tate [8]) that the Galois group $\text{Aut}(l_w/k_v)$ is isomorphic to the stabilizer

$$H_w = \{\sigma \in H : \sigma w = w\}.$$

As the completion of an archimedean local field is either \mathbb{R} or \mathbb{C} , it follows that each stabilizer H_w is either trivial, or is cyclic of order 2. More precisely, we have

$$|H_w| = 1 \quad \text{if and only if either } l_w \cong k_v \cong \mathbb{R} \text{ for all } w \text{ with } w|v, \\ \text{or } l_w \cong k_v \cong \mathbb{C} \text{ for all } w \text{ with } w|v,$$

and

$$|H_w| = 2 \quad \text{if and only if both } k_v \cong \mathbb{R} \text{ and } l_w \cong \mathbb{C} \text{ for all } w \text{ with } w|v.$$

Of course the same remark applies to the stabilizer

$$G_w = \{\sigma \in G : \sigma w = w\},$$

but now \mathbb{Q} has one archimedean place, and $\mathbb{Q}_\infty \cong \mathbb{R}$. We find that

$$(4.5) \quad |G_w| = 1 \quad \text{if and only if } l_w \cong \mathbb{R} \text{ for all } w \text{ with } w|\infty,$$

and

$$(4.6) \quad |G_w| = 2 \quad \text{if and only if } l_w \cong \mathbb{C} \text{ for all } w \text{ with } w|\infty.$$

Clearly (4.5) occurs when l/\mathbb{Q} is a totally real Galois extension, and (4.6) occurs when l/\mathbb{Q} is a totally complex Galois extension.

Let \hat{w} be a particular archimedean place of l . As before we write

$$G_{\hat{w}} = \{\sigma \in G : \sigma \hat{w} = \hat{w}\},$$

for the stabilizer of \hat{w} . We have $|G_{\hat{w}}| = [l_{\hat{w}} : \mathbb{Q}_{\hat{w}}]$, and therefore

$$|G_{\hat{w}}| = \begin{cases} 1 & \text{if } l/\mathbb{Q} \text{ is totally real,} \\ 2 & \text{if } l/\mathbb{Q} \text{ is totally complex.} \end{cases}$$

Write $[G : G_{\hat{w}}] = N$, so that N is the number of archimedean places of l . Let $\tau_1, \tau_2, \dots, \tau_N$ be a complete set of distinct representatives for the left cosets of the subgroup $G_{\hat{w}}$. Then

$$\{\tau_n \hat{w} : n = 1, 2, \dots, N\} = W_\infty(l/\mathbb{Q})$$

is a complete set of distinct archimedean places of l . To verify this, observe that if $\tau_m \hat{w} = \tau_n \hat{w}$, then $\tau_m^{-1} \tau_n$ belongs to $G_{\hat{w}}$ and it follows that τ_n is an element of the coset $\tau_m G_{\hat{w}}$. This is impossible because $\tau_1, \tau_2, \dots, \tau_N$ is a complete set of distinct representatives for the left cosets of $G_{\hat{w}}$. Therefore we have

$$(4.7) \quad \{ | \tau_n \hat{w} : n = 1, 2, \dots, N \} = \{ | w : w \in W_{\infty}(l/\mathbb{Q}) \}.$$

5. SPECIAL MINKOWSKI UNITS

In this section we assume that l/\mathbb{Q} is a finite Galois extension, and we write F_l for the free group (1.2) of positive rank. We say that β in F_l is a *special Minkowski unit* if there exists an archimedean place \hat{w} of l such that

$$(5.1) \quad \log |\beta|_w < 0 \quad \text{for all archimedean places } w \text{ such that } w \neq \hat{w}.$$

If β is a special Minkowski unit with respect to \hat{w} , then by the product formula we have

$$0 = \sum_{w|\infty} \log |\beta|_w = \log |\beta|_{\hat{w}} + \sum_{\substack{w|\infty \\ w \neq \hat{w}}} \log |\beta|_w,$$

and therefore

$$0 < \log |\beta|_{\hat{w}}.$$

If β_1 and β_2 are both special Minkowski units with respect to \hat{w} , then it is trivial that the product $\beta_1 \beta_2$ is also a special Minkowski unit with respect to \hat{w} . More generally, for each archimedean place \hat{w} we define

$$\mathcal{M}_{\hat{w}} = \{ \alpha \in F_l : \log |\alpha|_w < 0 \text{ for all } w|\infty \text{ such that } w \neq \hat{w} \},$$

so that $\mathcal{M}_{\hat{w}}$ is the set of all special Minkowski units with respect to \hat{w} . Then each subset $\mathcal{M}_{\hat{w}}$ is clearly a multiplicative semi-group in F_l . Later we will show that if $\mathfrak{A} \subseteq F_l$ is a subgroup of maximal rank, then

$$(5.2) \quad \mathfrak{A} \cap \mathcal{M}_{\hat{w}}$$

is not empty.

Lemma 5.1. *Let β be an element of F_l . Then the following conditions are equivalent.*

- (i) *The element β is a special Minkowski unit with respect to the archimedean place \hat{w} .*
- (ii) *If $T = \{\tau_1, \tau_2, \dots, \tau_N\}$ is a transversal for the left cosets of the subgroup $G_{\hat{w}}$, then*

$$(5.3) \quad \log |\tau_m^{-1} \tau_n(\beta)|_{\hat{w}} < 0, \quad \text{whenever } m \neq n.$$

Proof. Assume that (i) holds. If $m \neq n$ we have $\tau_m G_{\hat{w}} \neq \tau_n G_{\hat{w}}$, and therefore

$$w = (\tau_m^{-1} \tau_n)^{-1} \hat{w} = \tau_n^{-1} \tau_m \hat{w} \neq \hat{w}.$$

Then it follows from (5.1) that

$$\log |\tau_m^{-1} \tau_n(\beta)|_{\hat{w}} = \log |\beta|_w < 0,$$

which verifies (ii).

Assume that (ii) holds. We recall that G acts transitively on the collection $W_{\infty}(l/\mathbb{Q})$ of archimedean places of l . If η in G satisfies $\eta \hat{w} = w$, then we have

$$\{ \sigma \in G : \sigma \hat{w} = w \} = \eta G_{\hat{w}}.$$

As $\{\tau_1, \tau_2, \dots, \tau_N\}$ is a transversal for the left cosets of the subgroup $G_{\hat{w}}$, there exists a pair (m, n) such that

$$(\tau_m^{-1} \tau_n)^{-1} G_{\hat{w}} = \eta G_{\hat{w}}.$$

Therefore we have

$$(\tau_m^{-1} \tau_n)^{-1} \hat{w} = w.$$

If $m \neq n$, then $w \neq \hat{w}$, and (5.3) implies that

$$\log |\beta|_w = \log |\tau_m^{-1} \tau_n(\beta)|_{\hat{w}} < 0.$$

It follows that β is a special Minkowski unit with respect to \hat{w} . \square

The situation is further clarified by the following basic result.

Lemma 5.2. *Let β in F_l be a special Minkowski unit with respect to the archimedean place \hat{w} .*

- (i) *If $T = \{\tau_1, \tau_2, \dots, \tau_N\}$ is a transversal for the left cosets of the subgroup $G_{\hat{w}}$, then each subset of*

$$(5.4) \quad \{\tau_1(\beta), \tau_2(\beta), \dots, \tau_N(\beta)\}$$

with cardinality $N - 1$, is multiplicatively independent in F_l .

- (ii) *The number β is a Minkowski unit.*

Proof. Define the $N \times N$ real matrix

$$(5.5) \quad M(\beta, T, \hat{w}) = (\log |\tau_m^{-1} \tau_n(\beta)|_{\hat{w}}),$$

where $m = 1, 2, \dots, N$ indexes rows, and $n = 1, 2, \dots, N$ indexes columns. Because T is a transversal for the left cosets of $G_{\hat{w}}$, we have

$$\{\tau_m \hat{w} : m = 1, 2, \dots, N\} = W_{\infty}(l/\mathbb{Q}).$$

Therefore the matrix $M(\beta, T, \hat{w})$ satisfies

$$(5.6) \quad \begin{aligned} \sum_{m=1}^N \log |\tau_m^{-1} \tau_n(\beta)|_{\hat{w}} &= \sum_{m=1}^N \log |\tau_n(\beta)|_{\tau_m \hat{w}} \\ &= \sum_{w|_{\infty}} \log |\tau_n(\beta)|_w \\ &= 0 \end{aligned}$$

by the product formula. If $m \neq n$, then it follows from Lemma 5.2 that

$$(5.7) \quad \log |\tau_m^{-1} \tau_n(\beta)|_{\hat{w}} < 0.$$

The identity (5.6) and the inequality (5.7) verify the hypotheses (2.16) and (2.17) in the statement of Lemma 2.4. We conclude that the \mathbb{Q} -rank and the \mathbb{R} -rank of $M(\beta, T, \hat{w})$ are both equal to $N - 1$, and each $(N - 1) \times (N - 1)$ submatrix of $M(\beta, T, \hat{w})$ is nonsingular. The set of columns of the matrix $M(\beta, T, \hat{w})$ is

$$\{(\log |\tau_m^{-1} \tau_n(\beta)|_{\hat{w}}) : n = 1, 2, \dots, N\} = \{(\log |\tau_n(\beta)|_w) : n = 1, 2, \dots, N\},$$

and we conclude that each subset of $N - 1$ distinct columns is \mathbb{Q} -linearly independent. This clearly implies the conclusion (i) in the statement of the lemma, and (ii) follows immediately. \square

If the Galois extension l/\mathbb{Q} is totally real, then the subgroup $G_{\hat{w}}$ is trivial, and the transversal T that occurs in the proof of Lemma 5.1 is the group G . In this case we have the identity (5.6) for column sums, and the corresponding identity

$$(5.8) \quad \begin{aligned} \sum_{n=1}^N \log |\tau_m^{-1} \tau_n(\beta)|_{\hat{w}} &= \sum_{n=1}^N \log |\tau_n(\beta)|_{\tau_m \hat{w}} \\ &= \log |\text{norm}_{l/\mathbb{Q}}(\beta)|_{\tau_m \hat{w}} \\ &= 0 \end{aligned}$$

for row sums. It will be useful to work in an analogous situation when l/\mathbb{Q} is a totally complex Galois extension.

Lemma 5.3. *Let l/\mathbb{Q} be a totally complex Galois extension, \hat{w} an archimedean place of l , and let β be a special Minkowski unit with respect to \hat{w} . Write*

$$(5.9) \quad G_{\hat{w}} = \{1, \rho\}, \quad \text{where } \rho^2 = 1.$$

Then both $\rho(\beta)$ and $\beta\rho(\beta)$ are special Minkowski units with respect to \hat{w} . Moreover, if $T = \{\tau_1, \tau_2, \dots, \tau_N\}$ is a transversal for the left cosets of the subgroup $G_{\hat{w}}$, then we have

$$(5.10) \quad \sum_{n=1}^N \log |\tau_m^{-1} \tau_n(\beta\rho(\beta))|_{\hat{w}} = 0$$

for each $m = 1, 2, \dots, N$.

Proof. We define the $N \times N$ real matrix

$$M(\beta, T, \hat{w}) = (\log |\tau_m^{-1} \tau_n(\beta)|_{\hat{w}}),$$

where $m = 1, 2, \dots, N$ indexes rows, and $n = 1, 2, \dots, N$ indexes columns. As in our proof of Lemma 5.2, the matrix $M(\beta, T, \hat{w})$ satisfies (5.6) and (5.7). Using (5.9) we find that

$$T\rho = \{\tau_1\rho, \tau_2\rho, \dots, \tau_N\rho\}$$

is a second transversal for the left cosets of $G_{\hat{w}}$. Hence the matrix

$$M(\beta, T\rho, \hat{w}) = (\log |(\tau_m\rho)^{-1} \tau_n\rho(\beta)|_{\hat{w}}),$$

where $m = 1, 2, \dots, N$ indexes rows and $n = 1, 2, \dots, N$ indexes columns, also satisfies the identity

$$(5.11) \quad \begin{aligned} \sum_{m=1}^N \log |(\tau_m\rho)^{-1} \tau_n\rho(\beta)|_{\hat{w}} &= \sum_{m=1}^N \log |\tau_n\rho(\beta)|_{\tau_m\rho\hat{w}} \\ &= \sum_{w|\infty} \log |\tau_n\rho(\beta)|_w \\ &= 0, \end{aligned}$$

and the inequality

$$\log |(\tau_m\rho)^{-1} \tau_n\rho(\beta)|_{\hat{w}} < 0.$$

Because ρ belongs to the stabilizer $G_{\hat{w}}$, we have $\rho\hat{w} = \hat{w}$. Therefore the (m, n) entry in the matrix $M(\beta, T\rho, \hat{w})$ is

$$\begin{aligned}
 \log |(\tau_m \rho)^{-1} \tau_n \rho(\beta)|_{\hat{w}} &= \log |\tau_n \rho(\beta)|_{\tau_m \rho \hat{w}} \\
 (5.12) \qquad \qquad \qquad &= \log |\tau_n \rho(\beta)|_{\tau_m \hat{w}} \\
 &= \log |\tau_m^{-1} \tau_n (\rho(\beta))|_{\hat{w}}.
 \end{aligned}$$

The identity (5.12) implies that

$$M(\beta, T\rho, \hat{w}) = M(\rho(\beta), T, \hat{w}).$$

As T is an arbitrary left transversal for $G_{\hat{w}}$, it follows from Lemma 5.2 that $\rho(\beta)$ is a special Minkowski unit with respect to the place \hat{w} . We have shown that both β and $\rho(\beta)$ belong to the semigroup $\mathcal{M}_{\hat{w}}$ of special Minkowski units. Therefore the product $\beta\rho(\beta)$ is a special Minkowski unit.

Because

$$G = \{\tau_1, \tau_2, \dots, \tau_N\} \cup \{\tau_1 \rho, \tau_2 \rho, \dots, \tau_N \rho\} = T \cup T\rho,$$

we find that

$$\begin{aligned}
 \sum_{n=1}^N \log |\tau_m^{-1} \tau_n (\beta\rho(\beta))|_{\hat{w}} &= \sum_{n=1}^N \log |\tau_n (\beta\rho(\beta))|_{\tau_m \hat{w}} \\
 &= \sum_{n=1}^N \log |\tau_n (\beta)|_{\tau_m \hat{w}} + \sum_{n=1}^N \log |\tau_n \rho(\beta)|_{\tau_m \hat{w}} \\
 &= \log |\text{norm}_{l/\mathbb{Q}}(\beta)|_{\tau_m \hat{w}} \\
 &= 0.
 \end{aligned}$$

This proves (5.10). \square

Let $\mathfrak{A} \subseteq F_l$ be a subgroup with maximal rank, and let \hat{w} be an archimedean place of l . We now prove that the subgroup \mathfrak{A} contains a special Minkowski unit β with respect to the place \hat{w} . It follows from Lemma 5.2 that β is a Minkowski unit in \mathfrak{A} . We construct β so that the Weil height of β is comparable with the sum of the heights of a basis for the subgroup \mathfrak{A} . We also give a bound on the index of the subgroup generated by the conjugates of β in the full group of units F_l .

Theorem 5.1. *Let l/\mathbb{Q} be a Galois extension with N archimedean places, and let \hat{w} be a particular archimedean place of l . Let $\eta_1, \eta_2, \dots, \eta_{N-1}$ be multiplicatively independent units in F_l , and write*

$$\mathfrak{A} = \langle \eta_1, \eta_2, \dots, \eta_{N-1} \rangle \subseteq F_l$$

for the subgroup of rank $N - 1$ that they generate. Then there exists a special Minkowski unit β with respect to \hat{w} that satisfies the following conditions.

- (i) *The unit β belongs to \mathfrak{A} .*
- (ii) *The height of β is bounded by*

$$(5.13) \qquad h(\beta) \leq 2 \sum_{n=1}^{N-1} h(\eta_n).$$

- (iii) If $T = \{\tau_1, \tau_2, \dots, \tau_N\}$ is a transversal for the left cosets of the subgroup $G_{\widehat{w}}$, then the $N \times N$ matrix

$$(5.14) \quad M(\beta, T, \widehat{w}) = (\log |\tau_m^{-1} \tau_n(\beta)|_{\widehat{w}}),$$

where $m = 1, 2, \dots, N$ indexes rows and $n = 1, 2, \dots, N$ indexes columns, has \mathbb{Q} -rank and \mathbb{R} -rank equal to $N - 1$.

- (iv) Each $(N - 1) \times (N - 1)$ submatrix of $M(\beta, T, \widehat{w})$ is nonsingular.
 (v) If $\mathbf{y} \neq \mathbf{0}$ is a point in \mathbb{R}^N such that

$$0 = \sum_{n=1}^N y_n \log |\tau_n(\beta)|_w, \quad \text{for each place } w \text{ in } W_\infty(l/\mathbb{Q}),$$

then the co-ordinates y_1, y_2, \dots, y_N are all positive, or all negative.

- (vi) The subgroup

$$\mathfrak{B} = \langle \tau_1(\beta), \tau_2(\beta), \dots, \tau_N(\beta) \rangle \subseteq F_l,$$

generated by the conjugate units has rank $N - 1$, and index bounded by

$$(5.15) \quad \text{Reg}(l)[F_l : \mathfrak{B}] \leq ([l : \mathbb{Q}]h(\beta))^{N-1},$$

where $\text{Reg}(l)$ is the regulator of l .

Proof. Let

$$A = (\log |\eta_n|_w)$$

be the $(N - 1) \times (N - 1)$ real matrix, where $w|_\infty$ with $w \neq \widehat{w}$ indexes rows, and $n = 1, 2, \dots, N - 1$ indexes columns. By Lemma 2.1 there exists a point ξ in \mathbb{Z}^{N-1} such that

$$(5.16) \quad 0 < \sum_{n=1}^{N-1} \xi_n \log |\eta_n|_w \leq \sum_{n=1}^{N-1} |\log |\eta_n|_w|$$

for each archimedean place w of l with $w \neq \widehat{w}$. Then it is obvious that $\xi \neq \mathbf{0}$. Let

$$\beta^{-1} = \prod_{n=1}^{N-1} \eta_n^{\xi_n},$$

so that $\beta \neq 1$, and β is an element of the subgroup \mathfrak{A} . In view of (5.16) we have

$$(5.17) \quad - \sum_{n=1}^{N-1} |\log |\eta_n|_w| \leq - \sum_{n=1}^{N-1} \xi_n \log |\eta_n|_w = \log |\beta|_w < 0$$

at each archimedean place w of l with $w \neq \widehat{w}$. From the product formula we get

$$(5.18) \quad 0 < \log |\beta|_{\widehat{w}} = - \sum_{\substack{w|_\infty \\ w \neq \widehat{w}}} \log |\beta|_w \leq \sum_{n=1}^{N-1} \sum_{\substack{w|_\infty \\ w \neq \widehat{w}}} |\log |\eta_n|_w|.$$

This leads to the estimate

$$\begin{aligned}
 2h(\beta) &= |\log |\beta|_{\widehat{w}}| + \sum_{\substack{w|\infty \\ w \neq \widehat{w}}} |\log |\beta|_w| \\
 &= \log |\beta|_{\widehat{w}} - \sum_{\substack{w|\infty \\ w \neq \widehat{w}}} \log |\beta|_w \\
 (5.19) \quad &\leq 2 \sum_{n=1}^{N-1} \sum_{\substack{w|\infty \\ w \neq \widehat{w}}} |\log |\eta_n|_w| \\
 &\leq 4 \sum_{n=1}^{N-1} h(\eta_n),
 \end{aligned}$$

which verifies (5.13).

Because of the identity (4.4) we have

$$(5.20) \quad M(\beta, T, \widehat{w}) = (\log |\tau_n(\beta)|_{\tau_n \widehat{w}}) = (\log |\tau_m^{-1}(\tau_n(\beta))|_{\widehat{w}}),$$

where $m = 1, 2, \dots, N$ indexes rows and $n = 1, 2, \dots, N$ indexes columns. The identity (5.20) determines an ordering for the archimedean places w that index the rows of $M(\beta, T, \widehat{w})$. But the choice of ordering does not effect the rank of $M(\beta, T, \widehat{w})$. If $m \neq n$ then

$$(\tau_m^{-1} \tau_n)^{-1} = \tau_n^{-1} \tau_m$$

is *not* in the subgroup $G_{\widehat{w}}$ that fixes \widehat{w} . It follows from (5.17) that

$$(5.21) \quad \log |\tau_m^{-1}(\tau_n(\beta))|_{\widehat{w}} < 0$$

whenever $m \neq n$. We also get

$$\begin{aligned}
 \sum_{m=1}^N \log |\tau_m^{-1}(\tau_n(\beta))|_{\widehat{w}} &= \sum_{m=1}^N \log |\tau_n(\beta)|_{\tau_m \widehat{w}} \\
 (5.22) \quad &= \sum_{w|\infty} \log |\tau_n(\beta)|_w \\
 &= 0,
 \end{aligned}$$

by appealing to (4.7) and the product formula. It follows from (5.21) and (5.22) that the matrix $M(\beta, T, \widehat{w})$ satisfies the hypotheses of Lemma 2.4. Hence $M(\beta, T, \widehat{w})$ also satisfies the conclusions of Lemma 2.4, and this verifies (iii), (iv), and (v).

We have shown that the column vectors

$$\{(\log |\tau_n(\beta)|_w) : n = 1, 2, \dots, N\} \subseteq \mathbb{R}^N$$

generate a subgroup of rank $N - 1$. Hence the subgroup

$$\mathfrak{B} = \langle \tau_n(\beta) : n = 1, 2, \dots, N \rangle \subseteq F_l,$$

generated by their inverse image in F_l , also has rank $N - 1$. Then the inequality (5.15) follows from [1, Theorem 1.1], and the observation that the map

$$n \mapsto h(\tau_n(\beta))$$

is constant. □

If N is the number of archimedean places of the Galois extension l/\mathbb{Q} , then the rank of the group F_l is $r(l) = N - 1$. Hence Theorem 1.1 follows immediately from (i), (ii), and (vi), in the statement of Theorem 5.1.

6. RELATIVE UNITS I: DEFINITIONS

Throughout this section we suppose that k and l are algebraic number fields with

$$\mathbb{Q} \subseteq k \subseteq l.$$

We write $r(k)$ for the rank of the unit group O_k^\times , and $r(l)$ for the rank of the unit group O_l^\times . Then k has $r(k) + 1$ archimedean places, and l has $r(l) + 1$ archimedean places. In general we have $r(k) \leq r(l)$, and we recall (see [7, Proposition 3.20]) that $r(k) = r(l)$ if and only if l is a CM-field, and k is the maximal totally real subfield of l .

The norm is a homomorphism of multiplicative groups

$$\text{Norm}_{l/k} : l^\times \rightarrow k^\times.$$

If v is a place of k , then each element α in l^\times satisfies the identity

$$(6.1) \quad [l : k] \sum_{w|v} \log |\alpha|_w = \log |\text{Norm}_{l/k}(\alpha)|_v,$$

where the absolute values $|\cdot|_v$ and $|\cdot|_w$ are normalized as in (4.1). It follows from (6.1) that the norm, restricted to the subgroup O_l^\times of units, is a homomorphism

$$\text{Norm}_{l/k} : O_l^\times \rightarrow O_k^\times,$$

and the norm, restricted to the torsion subgroup in O_l^\times , is also a homomorphism

$$\text{Norm}_{l/k} : \text{Tor}(O_l^\times) \rightarrow \text{Tor}(O_k^\times).$$

Therefore we get a well defined homomorphism, which we write as

$$\text{norm}_{l/k} : O_l^\times / \text{Tor}(O_l^\times) \rightarrow O_k^\times / \text{Tor}(O_k^\times),$$

and define by

$$(6.2) \quad \text{norm}_{l/k}(\alpha \text{Tor}(O_l^\times)) = \text{Norm}_{l/k}(\alpha) \text{Tor}(O_k^\times).$$

However, to simplify notation we write

$$(6.3) \quad F_k = O_k^\times / \text{Tor}(O_k^\times), \quad \text{and} \quad F_l = O_l^\times / \text{Tor}(O_l^\times),$$

so that

$$(6.4) \quad \text{norm}_{l/k} : F_l \rightarrow F_k.$$

We also write the elements of the quotient groups F_k and F_l as coset representatives, rather than as cosets. And by abuse of language, we continue to refer to the elements of F_k and F_l as units. Obviously F_k and F_l are free abelian groups of rank $r(k)$ and $r(l)$, respectively. As

$$O_k^\times \subseteq O_l^\times,$$

we can identify F_k with the subgroup

$$O_k^\times / \text{Tor}(O_l^\times),$$

and in this way regard F_k as a subgroup of F_l . We also note that (6.1) and (6.4) imply that

$$(6.5) \quad [l : k] \sum_{w|v} \log |\alpha|_w = \log |\text{norm}_{l/k}(\alpha)|_v$$

for each place v of k and each point α in F_l .

Following Costa and Friedman [4] and [5], the subgroup of relative units in O_l^\times is defined by

$$\{\alpha \in O_l^\times : \text{Norm}_{l/k}(\alpha) \in \text{Tor}(O_k^\times)\}.$$

Here we work in the free group F_l where the image of the subgroup of relative units is the kernel of the homomorphism $\text{norm}_{l/k}$. Therefore we define the subgroup of *relative units* in F_l to be the subgroup

$$(6.6) \quad E_{l/k} = \{\alpha \in F_l : \text{norm}_{l/k}(\alpha) = 1\}.$$

We also write

$$(6.7) \quad I_{l/k} = \{\text{norm}_{l/k}(\alpha) : \alpha \in F_l\} \subseteq F_k$$

for the image of the homomorphism $\text{norm}_{l/k}$. If β in F_l represents a coset in the subgroup F_k , then we have

$$\text{norm}_{l/k}(\beta) = \beta^{[l:k]}.$$

Therefore the image $I_{l/k} \subseteq F_k$ is a subgroup of rank $r(k)$, and the index satisfies

$$(6.8) \quad [F_k : I_{l/k}] < \infty.$$

It follows that $E_{l/k} \subseteq F_l$ is a subgroup of rank $r(l/k) = r(l) - r(k)$. We restrict our attention to extensions l/k such that

$$(6.9) \quad 1 \leq r(k) < r(l).$$

The inequality $1 \leq r(k)$ implies that k is not \mathbb{Q} , and k is not an imaginary, quadratic extension of \mathbb{Q} . And the inequality $r(k) < r(l)$ implies that l is not a CM-field such that k is the maximal totally real subfield of l . Alternatively, the hypothesis (6.9) implies that $E_{l/k}$ is a proper subgroup of F_l . As F_k is a free group, it follows that the kernel $E_{l/k}$ of the homomorphism (6.4) is a direct sum in F_l .

7. RELATIVE UNITS II: H IS NORMAL IN G

We continue to assume that

$$(7.1) \quad \mathbb{Q} \subseteq k \subseteq l,$$

that these fields satisfy the inequality (6.9), and we also assume that both l/\mathbb{Q} and k/\mathbb{Q} are finite, Galois extensions. We write

$$H = \text{Aut}(l/k), \quad \text{and} \quad G = \text{Aut}(l/\mathbb{Q}),$$

but now H is a normal subgroup of G . Therefore we have the canonical homomorphism

$$(7.2) \quad \varphi : G \rightarrow G/H,$$

and the isomorphism

$$(7.3) \quad G/H \cong \text{Aut}(k/\mathbb{Q}).$$

The isomorphism (7.3) is the map that restricts the domain of a coset representative in G/H to the subfield k .

Lemma 7.1. *Assume that the number fields (7.1) are such that both l/\mathbb{Q} and k/\mathbb{Q} are finite, Galois extensions. Then for each place u of \mathbb{Q} the map*

$$(7.4) \quad v \mapsto |W_v(l/k)|$$

is constant on the collection of places v in $W_u(k/\mathbb{Q})$, and the group G acts transitively on the collection of disjoint subsets

$$(7.5) \quad \{W_v(l/k) : v \in W_u(k/\mathbb{Q})\}.$$

Moreover, for each automorphism τ in G and each place v of k , this action satisfies the identity

$$(7.6) \quad \tau W_v(l/k) = W_{\eta v}(l/k),$$

where φ is the canonical homomorphism (7.2) and $\eta = \varphi(\tau)$.

Proof. Let u be a fixed place of \mathbb{Q} . Then for each place v in $W_u(k/\mathbb{Q})$ and each place w in $W_v(l/k)$, we have

$$(7.7) \quad [l_w : \mathbb{Q}_u] = [l_w : k_v][k_v : \mathbb{Q}_u].$$

As l/\mathbb{Q} is Galois, the map

$$w \mapsto [l_w : \mathbb{Q}_u]$$

is constant for w in $W_u(l/\mathbb{Q})$. Similarly, the extension k/\mathbb{Q} is Galois and therefore the map

$$v \mapsto [k_v : \mathbb{Q}_u]$$

is constant for v in $W_u(k/\mathbb{Q})$. It follows from these observations and the identity (7.7), that

$$(7.8) \quad (v, w) \mapsto [l_w : k_v]$$

is constant for pairs (v, w) such that v belongs to $W_u(k/\mathbb{Q})$ and w belongs to $W_v(l/k)$. Next we recall that the global degree of the extension l/k is the sum of local degrees, so that for each place v in $W_u(l/k)$ we have

$$(7.9) \quad [l : k] = \sum_{w|v} [l_w : k_v].$$

Then using (7.8) we get the identity

$$(7.10) \quad [l : k] = [l_w : k_v] |W_v(l/k)|$$

for each pair (v, w) such that v belongs to $W_u(k/\mathbb{Q})$ and w belongs to $W_v(l/k)$. Now (7.8) and (7.10) imply that the map (7.4) is constant for v in $W_u(k/\mathbb{Q})$.

If u is a place of \mathbb{Q} , we have the disjoint union

$$(7.11) \quad W_u(l/\mathbb{Q}) = \bigcup_{v|u} W_v(l/k),$$

where the union on the right of (7.11) is over the collection of places v in $W_u(k/\mathbb{Q})$. Let τ belong to G and let $\varphi(\tau) = \eta$, where η is an automorphism in $\text{Aut}(k/\mathbb{Q})$, and let w be a place in $W_v(l/k)$. Then for each point β in k we have

$$\|\beta\|_w = \|\beta\|_v,$$

and, as k/\mathbb{Q} is Galois, we have

$$\tau^{-1}(\beta) = \eta^{-1}(\beta)$$

in k . Therefore, we get

$$(7.12) \quad \|\beta\|_{\tau w} = \|\tau^{-1}(\beta)\|_w = \|\eta^{-1}(\beta)\|_v = \|\beta\|_{\eta v}$$

at each point β in k . The identity (7.12) implies that $\tau w | \eta v$. As w in $W_v(l/k)$ was arbitrary, we have

$$(7.13) \quad \tau W_v(l/k) \subseteq W_{\eta v}(l/k).$$

It follows from (7.4) that $W_v(l/k)$ and $W_{\eta v}(l/k)$ have the same cardinality. And it is trivial to check that $\tau W_v(l/k)$ and $W_v(l/k)$ have the same cardinality. Hence there is equality in the inclusion (7.13). This shows that G acts transitively on the collection of disjoint subsets (7.5), and also establishes the identity (7.6). \square

We recall from (6.6) that

$$(7.14) \quad \begin{aligned} E_{l/k} &= \{\alpha \in F_l : \text{norm}_{l/k}(\alpha) = 1\} \\ &= \left\{ \alpha \in F_l : \sum_{w|v} \log |\alpha|_w = 0 \text{ for each place } v \text{ in } W_\infty(k/\mathbb{Q}) \right\}. \end{aligned}$$

Because l/\mathbb{Q} is Galois, the subextension l/k is Galois, and the group H acts transitively on each subset $W_v(l/k)$. Then it follows from (6.5) and (7.14) that H acts on $E_{l/k}$. Here we also assume that k/\mathbb{Q} is a Galois extension. We now show that this additional hypothesis implies that the group G acts on $E_{l/k}$.

Lemma 7.2. *Assume that the number fields (7.1) are such that both l/\mathbb{Q} and k/\mathbb{Q} are finite, Galois extensions. Then the group $G = \text{Aut}(l/\mathbb{Q})$ acts on the subgroup $E_{l/k}$ of relative units.*

Proof. As G acts on elements of the group F_l , it suffices to show that if α belongs to the subgroup $E_{l/k} \subseteq F_l$ then $\tau(\alpha)$ belongs to $E_{l/k}$ for each automorphism τ in G . Therefore we suppose that τ belongs to G , and we let $\varphi(\tau^{-1}) = \eta$, where φ is the canonical homomorphism (7.2). If α belongs to $E_{l/k}$, then using the identity (7.6) in the statement of Lemma 7.1 we find that

$$(7.15) \quad \begin{aligned} \sum_{w|v} \log |\tau(\alpha)|_w &= \sum_{w|v} \log |\alpha|_{\tau^{-1}w} \\ &= \sum_{w \in \tau^{-1}W_v(l/k)} \log |\alpha|_w \\ &= \sum_{w|\eta v} \log |\alpha|_w \\ &= 0. \end{aligned}$$

We conclude that $\tau(\alpha)$ belongs to $E_{l/k}$. \square

It follows from Lemma 7.2 that an element $\alpha \neq 1$ in the group $E_{l/k}$ has an orbit

$$(7.16) \quad \{\tau(\alpha) : \tau \in G\} \subseteq E_{l/k}.$$

If the subset on the left of (7.16) contains $r(l/k)$ multiplicatively independent elements, then we say that $\alpha \neq 1$ is a *relative Minkowski unit* for the subgroup $E_{l/k}$. If k is \mathbb{Q} , or if k is an imaginary, quadratic extension of \mathbb{Q} , then $r(k) = 0$, and $E_{l/k} = F_l$. In this case $\alpha \neq 1$ in $E_{l/k}$ is a relative Minkowski unit for $E_{l/k}$ if and only if $\alpha \neq 1$ is a Minkowski unit for F_l . On the other hand, if l is a CM-field,

and k is the maximal totally real subfield of l , then $r(l/k) = r(l) - r(k) = 0$, the subgroup $E_{l/k}$ is trivial, and relative Minkowski units do not exist.

8. RELATIVE UNITS III: A BI-HOMOMORPHISM

In this section we continue to assume that l/\mathbb{Q} and k/\mathbb{Q} are both Galois extensions, or equivalently that H is a normal subgroup of G . We define a bi-homomorphism

$$(8.1) \quad \Delta : F_l \times \mathbb{Z}^N \rightarrow F_l,$$

where N is the cardinality of $W_\infty(l/\mathbb{Q})$. To define the bi-homomorphism (8.1) we first select an archimedean place \widehat{w} in $W_\infty(l/\mathbb{Q})$, and a transversal

$$(8.2) \quad \Psi = \{\psi_1, \psi_2, \dots, \psi_N\}$$

for the left cosets of the stabilizer $G_{\widehat{w}}$. Thus we have the disjoint union

$$G = \bigcup_{n=1}^N \psi_n G_{\widehat{w}}.$$

The bi-homomorphism Δ depends on the choice of \widehat{w} and on the choice of the transversal (8.2), but to simplify notation we suppress this dependence. We write \mathbb{Z}^N for the free abelian group of rank N , and we identify the elements of this group with functions

$$(8.3) \quad f : G \rightarrow \mathbb{Z},$$

that are constant on left cosets of $G_{\widehat{w}}$. We recall that $[G : G_{\widehat{w}}] = N$, so that the group of such functions does form a free abelian group of rank N . If $G_{\widehat{w}}$ is not trivial, then it has order 2, and

$$G_{\widehat{w}} = \{1, \rho\}, \quad \text{where } \rho^2 = 1.$$

In this case each left coset of $G_{\widehat{w}}$ has two representatives. We find that

$$\eta G_{\widehat{w}} = \{\eta, \eta\rho\} = \eta\rho\{1, \rho\} = \eta\rho G_{\widehat{w}}.$$

Thus a function f as in (8.3), belongs to \mathbb{Z}^N if and only if it satisfies the identity

$$(8.4) \quad f(\eta) = f(\eta\rho)$$

for each element η in G . If (α, f) is an element of the product $F_l \times \mathbb{Z}^N$, we define

$$\Delta(\alpha, f) = \prod_{n=1}^N \psi_n(\alpha)^{f(\psi_n)}.$$

As G acts on the group F_l , it is obvious that $\Delta(\alpha, f)$ belongs to F_l . If (α_1, f) and (α_2, f) are both elements of $F_l \times \mathbb{Z}^N$, we find that

$$(8.5) \quad \begin{aligned} \Delta(\alpha_1, f)\Delta(\alpha_2, f) &= \left(\prod_{m=1}^N \psi_m(\alpha_1)^{f(\psi_m)} \right) \left(\prod_{n=1}^N \psi_n(\alpha_2)^{f(\psi_n)} \right) \\ &= \prod_{n=1}^N \psi_n(\alpha_1\alpha_2)^{f(\psi_n)} = \Delta(\alpha_1\alpha_2, f). \end{aligned}$$

This shows that for each fixed element f in \mathbb{Z}^N , the map

$$\alpha \mapsto \Delta(\alpha, f)$$

is a homomorphism from F_l into F_l . In a similar manner, if (α, f_1) and (α, f_2) are both elements of $F_l \times \mathbb{Z}^N$, we get

$$(8.6) \quad \Delta(\alpha, f_1)\Delta(\alpha, f_2) = \Delta(\alpha, f_1 + f_2).$$

The identity (8.6) shows that for each fixed α in F_l , the map

$$(8.7) \quad f \mapsto \Delta(\alpha, f)$$

is a homomorphism from \mathbb{Z}^N into F_l . Hence the map Δ is a bi-homomorphism.

It will be convenient to use the $\|\cdot\|_1$ -norm on functions f in \mathbb{Z}^N . Therefore we set

$$\|f\|_1 = \sum_{n=1}^N |f(\psi_n)|$$

for each f in \mathbb{Z}^N .

Lemma 8.1. *If (α, f) is an element of $F_l \times \mathbb{Z}^N$, then*

$$(8.8) \quad h(\Delta(\alpha, f)) \leq \|f\|_1 h(\alpha).$$

Proof. The map

$$n \mapsto h(\psi_n(\alpha))$$

is constant. Thus we have

$$\begin{aligned} 2h(\Delta(\alpha, f)) &= \sum_{w|\infty} |\log |\Delta(\alpha, f)|_w| \\ &= \sum_{w|\infty} \left| \sum_{n=1}^N f(\psi_n) \log |\psi_n(\alpha)|_w \right| \\ &\leq \sum_{n=1}^N |f(\psi_n)| \sum_{w|\infty} |\log |\psi_n(\alpha)|_w| \\ &= 2 \sum_{n=1}^N |f(\psi_n)| h(\psi_n(\alpha)) \\ &= 2\|f\|_1 h(\alpha). \end{aligned}$$

This proves the lemma. \square

We have noted that the group G acts on F_l . The group G also acts on \mathbb{Z}^N . More precisely, if η belongs to G and $x \mapsto f(x)$ belongs to \mathbb{Z}^N , we denote the action of η on f by $[\eta, f]$, where

$$(8.9) \quad [\eta, f](x) = f(\eta^{-1}x).$$

If f satisfies

$$f(\tau) = f(\tau\rho)$$

for each τ in G , then it is obvious that

$$(8.10) \quad f(\eta^{-1}\tau) = f(\eta^{-1}\tau\rho)$$

for each τ in G . That is, if η belongs to G and f belongs to \mathbb{Z}^N , then $[\eta, f]$ belongs to \mathbb{Z}^N . It is clear that

$$(8.11) \quad [1, f](x) = f(x).$$

If both η_1 and η_2 belong to G , then

$$\begin{aligned}
 (8.12) \quad [\eta_1, [\eta_2, f]](x) &= [\eta_2, f](\eta_1^{-1}x) \\
 &= f(\eta_2^{-1}\eta_1^{-1}x) \\
 &= f((\eta_1\eta_2)^{-1}x) \\
 &= [\eta_1\eta_2, f](x).
 \end{aligned}$$

The identities (8.10), (8.11), and (8.12), verify that (8.9) defines an action of the group G on the collection of functions in \mathbb{Z}^N .

The action of G on the \mathbb{Z}^N occurs in the following identities.

Lemma 8.2. *Let η belong to G , and let (α, f) be a point in $F_l \times \mathbb{Z}^N$. If l/\mathbb{Q} is a totally real Galois extension, then $G_{\widehat{w}}$ is trivial and*

$$(8.13) \quad \eta(\Delta(\alpha, f)) = \Delta(\alpha, [\eta, f]).$$

If l/\mathbb{Q} is a totally complex Galois extension, then $G_{\widehat{w}}$ is cyclic of order 2,

$$(8.14) \quad G_{\widehat{w}} = \{1, \rho\}, \quad \text{where } \rho^2 = 1,$$

and

$$(8.15) \quad \eta(\Delta(\alpha\rho(\alpha), f)) = \Delta(\alpha\rho(\alpha), [\eta, f]).$$

Proof. We assume that l/\mathbb{Q} totally real. Then we have

$$G = \{\psi_1, \psi_2, \dots, \psi_N\},$$

and

$$\begin{aligned}
 (8.16) \quad \eta(\Delta(\alpha, f)) &= \prod_{n=1}^N (\eta\psi_n(\alpha))^{f(\psi_n)} \\
 &= \prod_{n=1}^N \psi_n(\alpha)^{f(\eta^{-1}\psi_n)} \\
 &= \Delta(\alpha, [\eta, f]).
 \end{aligned}$$

Now assume that l/\mathbb{Q} is totally complex. Then

$$G = \{\psi_1, \psi_2, \dots, \psi_N\} \cup \{\psi_1\rho, \psi_2\rho, \dots, \psi_N\rho\} = \Psi \cup \Psi\rho,$$

and

$$\begin{aligned}
 (8.17) \quad \eta(\Delta(\alpha, f)\Delta(\rho(\alpha), f)) &= \prod_{m=1}^N (\eta\psi_m(\alpha))^{f(\psi_m)} \prod_{n=1}^N (\eta\psi_n\rho(\alpha))^{f(\psi_n\rho)} \\
 &= \prod_{m=1}^N (\psi_m(\alpha))^{f(\eta^{-1}\psi_m)} \prod_{n=1}^N (\psi_n\rho(\alpha))^{f(\eta^{-1}\psi_n\rho)} \\
 &= \Delta(\alpha, [\eta, f])\Delta(\rho(\alpha), [\eta, f]).
 \end{aligned}$$

When (8.5) and (8.17) are combined, we obtain the identity (8.15) for each automorphism η in G . \square

As F_l is a free abelian group of rank $N - 1$, it follows that for each α in F_l the kernel of the homomorphism (8.7) has rank greater than or equal to 1. The following result is an immediate consequence of Theorem 5.1

Theorem 8.1. *Let β in F_l be a special Minkowski unit with respect to \widehat{w} . Then the image*

$$\{\Delta(\beta, f) : f \in \mathbb{Z}^N\}$$

of the homomorphism (8.7) is a subgroup of F_l with rank $N - 1$.

Proof. It suffices to show that the kernel

$$\begin{aligned} \mathcal{K}(\beta) &= \{f \in \mathbb{Z}^N : \Delta(\beta, f) = 1\} \\ (8.18) \quad &= \{f \in \mathbb{Z}^N : \sum_{n=1}^N f(\psi_n) \log |\psi_n(\beta)|_w = 0 \text{ for each } w \text{ in } W_\infty(l/\mathbb{Q})\} \\ &= \{f \in \mathbb{Z}^N : \sum_{n=1}^N f(\psi_n) \log |\psi_n^{-1}(\beta)|_{\widehat{w}} = 0 \text{ for } m = 1, 2, \dots, N\} \end{aligned}$$

has rank 1. If we write f in \mathbb{Z}^N as a (column) vector \mathbf{f} , then $\mathcal{K}(\beta)$ is the null space of the linear transformation

$$\mathbf{f} \mapsto M(\beta, \Psi, \widehat{w})\mathbf{f},$$

where $M(\beta, \Psi, \widehat{w})$ is the $N \times N$ matrix defined in (5.14). As the matrix $M(\beta, \Psi, \widehat{w})$ has rank $N - 1$, it follows that the kernel $\mathcal{K}(\beta)$ has rank 1. \square

If f belongs to the kernel $\mathcal{K}(\beta)$ defined in (8.18) and f is not identically zero, then it follows from (v) in the statement of Theorem 5.1, that the co-ordinate function

$$n \mapsto f(\psi_n)$$

takes only positive values, or it takes only negative values. This will play a crucial role in our construction of relative Minkowski units.

We recall that H acts transitively on each collection of places $W_v(l/k)$, where v is a place in $W_\infty(k/\mathbb{Q})$, and G acts transitively on the collection of places $W_\infty(l/\mathbb{Q})$. Also by Lemma 7.1, the group G acts transitively on the collection of subsets

$$(8.19) \quad \{W_v(l/k) : v \in W_u(k/\mathbb{Q})\}.$$

If τ is an automorphism in G , and $\varphi : G \rightarrow G/H$ is the canonical homomorphism, then at each place v of k , the action of τ on the subsets in the collection (8.19) is given by

$$(8.20) \quad \tau W_v(l/k) = W_{\eta v}(l/k),$$

where $\eta = \varphi(\tau)$. If τ_1 and τ_2 belong to G , then it follows from (8.20) that

$$(8.21) \quad \tau_1 W_v(l/k) = \tau_2 W_v(l/k)$$

if and only if $\varphi(\tau_1) = \varphi(\tau_2)$. That is, (8.21) holds if and only if $\tau_1 H = \tau_2 H$.

There is a further implication of (8.20) that will be useful. Let α belong to F_l , and let v be a place in $W_\infty(k/\mathbb{Q})$. Then for each τ in G we have

$$\begin{aligned} \sum_{w|v} \log |\tau^{-1}(\alpha)|_w &= \sum_{w \in W_v(l/k)} \log |\alpha|_{\tau w} \\ &= \sum_{w \in \tau W_v(l/k)} \log |\alpha|_w \\ &= \sum_{w \in W_{\eta v}(l/k)} \log |\alpha|_w, \end{aligned}$$

where $\eta = \varphi(\tau)$. It follows that the map

$$(8.22) \quad \tau \mapsto \sum_{w|v} \log |\tau^{-1}(\alpha)|_w$$

from G into \mathbb{R} , depends only on $\eta = \varphi(\tau)$, and therefore (8.22) is constant for τ restricted to a coset of H . This observation was already used in the identity (7.15).

We select a place \widehat{w} in $W_\infty(l/\mathbb{Q})$. As before we write

$$G_{\widehat{w}} = \{\tau \in G : \tau \widehat{w} = \widehat{w}\},$$

for the stabilizer of \widehat{w} in G . We will continue to write

$$|G| = [l : \mathbb{Q}] = \begin{cases} N & \text{if } l/\mathbb{Q} \text{ is a totally real Galois extension,} \\ 2N & \text{if } l/\mathbb{Q} \text{ is a totally complex Galois extension,} \end{cases}$$

so that

$$[G : G_{\widehat{w}}] = |W_\infty(l/\mathbb{Q})| = N.$$

Let $I = [G : G_{\widehat{w}}H]$, and let

$$(8.23) \quad \{\tau_1, \tau_2, \dots, \tau_I\} \subseteq G$$

be a left transversal for the subgroup $G_{\widehat{w}}H$ in G . Then we have the disjoint union

$$(8.24) \quad G = \bigcup_{i=1}^I \tau_i G_{\widehat{w}}H.$$

Using Lemma 7.1, the rank of the group $E_{l/k}$ of relative units is given by

$$\begin{aligned} r(l/k) &= r(l) - r(k) \\ &= (|W_\infty(l/\mathbb{Q})| - 1) - (|W_\infty(k/\mathbb{Q})| - 1) \\ &= [G : G_{\widehat{w}}] - [G : G_{\widehat{w}}H] \\ &= N - I. \end{aligned}$$

Similarly, let $J = [G_{\widehat{w}}H : G_{\widehat{w}}]$, and let

$$(8.25) \quad \{\sigma_1, \sigma_2, \dots, \sigma_J\} \subseteq G_{\widehat{w}}H$$

be a left transversal for the subgroup $G_{\widehat{w}}$ in $G_{\widehat{w}}H$. Then we have the disjoint union

$$(8.26) \quad G_{\widehat{w}}H = \bigcup_{j=1}^J \sigma_j G_{\widehat{w}}.$$

If k/\mathbb{Q} is totally real then $G_{\widehat{w}}H = H$, and it is obvious that

$$(8.27) \quad \{\sigma_1, \sigma_2, \dots, \sigma_J\} = H.$$

If k/\mathbb{Q} is totally complex then $G_{\widehat{w}} = \{1, \rho\}$ has order 2, and $G_{\widehat{w}} \cap H$ is trivial. It follows that

$$G_{\widehat{w}}H = H \cup \rho H = H \cup H\rho, \quad \text{and} \quad [G_{\widehat{w}}H : G_{\widehat{w}}] = |H|.$$

In this case we select the transversal (8.25) so that

$$(8.28) \quad \{\sigma_1, \sigma_2, \dots, \sigma_J\} = H.$$

Combining (8.24) and (8.26), we find that

$$(8.29) \quad G = \bigcup_{i=1}^I \tau_i G_{\widehat{w}}H = \bigcup_{i=1}^I \tau_i \left(\bigcup_{j=1}^J \sigma_j G_{\widehat{w}} \right) = \bigcup_{i=1}^I \bigcup_{j=1}^J (\tau_i \sigma_j G_{\widehat{w}}).$$

It follows that

$$\{\tau_i \sigma_j : i = 1, 2, \dots, I, \text{ and } j = 1, 2, \dots, J\}$$

is a transversal for the subgroup $G_{\hat{w}}$ in G . We also have

$$N = [G : G_{\hat{w}}] = [G : G_{\hat{w}}H][G_{\hat{w}}H : G_{\hat{w}}] = IJ.$$

Next we require a variant of the fact that (8.22) depends only on $\eta = \varphi(\tau)$.

Lemma 8.3. *Let τ_i be a coset representative in (8.23), and let σ_j be a coset representative in (8.25). Let α be an element of the group F_l , and let v be a place in $W_\infty(k/\mathbb{Q})$. Then the map*

$$(8.30) \quad (\tau_i, \sigma_j) \mapsto \sum_{w|v} \log |\tau_i \sigma_j \alpha|_w$$

depends only on τ_i , and not on σ_j .

Proof. Let $\varphi : G \rightarrow G/H$ be the canonical homomorphism. Because $\text{Aut}(k/\mathbb{Q})$ acts transitively on $W_\infty(k/\mathbb{Q})$, and $\text{Aut}(k/\mathbb{Q})$ is isomorphic to G/H , there exists \hat{v} in $W_\infty(k/\mathbb{Q})$ so that

$$\varphi(\tau_i)\hat{v} = v, \quad \text{and} \quad \tau_i W_{\hat{v}}(l/k) = W_v(l/k).$$

Using the identity (7.6) in the statement of Lemma 7.1, we find that

$$(8.31) \quad \begin{aligned} \sum_{w|v} \log |\tau_i \sigma_j \alpha|_w &= \sum_{w \in W_v(l/k)} \log |\sigma_j \alpha|_{\tau_i^{-1}w} \\ &= \sum_{w \in \tau_i W_{\hat{v}}(l/k)} \log |\sigma_j \alpha|_{\tau_i^{-1}w} \\ &= \sum_{w|\hat{v}} \log |\sigma_j \alpha|_w. \end{aligned}$$

Because σ_j belongs to H , its image $\varphi(\sigma_j)$ is trivial. In this case (7.6) implies that

$$(8.32) \quad \sum_{w|\hat{v}} \log |\sigma_j \alpha|_w = \sum_{w|\hat{v}} \log |\alpha|_w.$$

Now (8.31) and (8.32) show that the map (8.30) depends on τ_i , but not on σ_j . \square

Let $\mathcal{L}_{l/k} \subseteq \mathbb{Z}^N$ be the subgroup

$$\mathcal{L}_{l/k} = \{f \in \mathbb{Z}^N : \sum_{j=1}^J f(\tau_i \sigma_j) = 0 \text{ for each } i = 1, 2, \dots, I\}.$$

The subsets

$$\tau_i G_{\hat{w}} H = \bigcup_{j=1}^J \tau_i \sigma_j G_{\hat{w}},$$

where $i = 1, 2, \dots, I$, are the distinct cosets of $G_{\hat{w}}H$ in G and are therefore disjoint. Hence the linear conditions defining the subgroup $\mathcal{L}_{l/k}$ are independent. It follows that

$$\text{rank } \mathcal{L}_{l/k} = N - I = r(l/k).$$

Theorem 8.2. *Let β in F_l be a special Minkowski unit. Then the image*

$$(8.33) \quad \{\Delta(\beta, f) : f \in \mathcal{L}_{l/k}\}$$

of the homomorphism (8.7) restricted to $\mathcal{L}_{l/k}$, is a subgroup of $E_{l/k}$ with rank $r(l/k)$.

Proof. Let v be a place in $W_\infty(k/\mathbb{Q})$, and let f belong to $\mathcal{L}_{l/k}$. Then we have

$$(8.34) \quad \begin{aligned} \sum_{w|v} \log |\Delta(\beta, f)|_w &= \sum_{w|v} \left(\sum_{i=1}^I \sum_{j=1}^J f(\tau_i \sigma_j) \log |\tau_i \sigma_j \beta|_w \right) \\ &= \sum_{i=1}^I \sum_{j=1}^J f(\tau_i \sigma_j) \left(\sum_{w|v} \log |\tau_i \sigma_j \beta|_w \right). \end{aligned}$$

By Lemma 8.3 the sum

$$\sum_{w|v} \log |\tau_i \sigma_j \beta|_w = c(\tau_i)$$

depends on τ_i , but not on σ_j . Hence on the right hand side of (8.34) we get

$$(8.35) \quad \sum_{j=1}^J f(\tau_i \sigma_j) \left(\sum_{w|v} \log |\tau_i \sigma_j \beta|_w \right) = c(\tau_i) \sum_{j=1}^J f(\tau_i \sigma_j) = 0$$

for each $i = 1, 2, \dots, I$. Combining (8.34) and (8.35), we conclude that

$$\sum_{w|v} \log |\Delta(\beta, f)|_w = 0$$

for each place v in $W_\infty(k/\mathbb{Q})$. We have shown that if f belongs to the subgroup $\mathcal{L}_{l/k}$, then $\Delta(\beta, f)$ belongs to the subgroup $E_{l/k}$.

Next we prove that the image (8.33) has rank $r(l/k)$. As $\mathcal{L}_{l/k}$ has rank $r(l/k)$, it suffices to show that the map $f \mapsto \Delta(\beta, f)$, restricted to the subgroup $\mathcal{L}_{l/k}$, is injective. This will follow if we show that its kernel is trivial. That is, it suffices to show that

$$(8.36) \quad \mathcal{K}(\beta) \cap \mathcal{L}_{l/k} = \{\mathbf{0}\},$$

where $\mathcal{K}(\beta)$ is defined by (8.18). We have already noted that if $f \neq \mathbf{0}$ belongs to $\mathcal{K}(\beta)$, then

$$n \mapsto f(\psi_n)$$

takes only positive values, or it takes only negative values. However, if $f \neq \mathbf{0}$ belongs to $\mathcal{L}_{l/k}$, then

$$\sum_{j=1}^J f(\tau_i \sigma_j) = 0$$

for each $i = 1, 2, \dots, I$. Therefore the only point in the intersection (8.36) is $\mathbf{0}$, and the theorem is proved. \square

As in the proof of Theorem 8.2, the map $f \mapsto \Delta(\beta, f)$ restricted to $\mathcal{L}_{l/k}$ is injective. Therefore we get the following result.

Corollary 8.1. *Let β in F_l be a special Minkowski unit, and let*

$$\{f_1, f_2, \dots, f_R\}, \quad \text{where } R = r(l/k),$$

be linearly independent elements in the free group $\mathcal{L}_{l/k}$. Then the elements of the set

$$\{\Delta(\beta, f_1), \Delta(\beta, f_2), \dots, \Delta(\beta, f_R)\}$$

are multiplicatively independent relative units in $E_{l/k}$.

9. RELATIVE UNITS IV: EXISTENCE

We continue to assume that l/\mathbb{Q} and k/\mathbb{Q} are finite, Galois extensions such that

$$\mathbb{Q} \subseteq k \subseteq l, \quad \text{and} \quad 1 \leq r(k) < r(l),$$

and we let \widehat{w} denote a particular archimedean place of l . If l/\mathbb{Q} is a totally complex Galois extension, then $G_{\widehat{w}}$ is cyclic of order 2, and we write

$$(9.1) \quad G_{\widehat{w}} = \{1, \rho\}, \quad \text{where } \rho^2 = 1.$$

In this section we apply Lemma 3.3 with

$$K = G_{\widehat{w}}, \quad H = \text{Aut}(l/k), \quad \text{and} \quad G = \text{Aut}(l/\mathbb{Q}),$$

and with λ in $\mathcal{L}_{l/k}$ defined by (3.10). Let $\{\tau_1, \tau_2, \dots, \tau_I\}$ be a transversal for the left cosets of $G_{\widehat{w}}H$ in G , and let $\{\sigma_1, \sigma_2, \dots, \sigma_J\}$ be a transversal for the left cosets of $G_{\widehat{w}}$ in $G_{\widehat{w}}H$. For each $i = 1, 2, \dots, I$, let

$$\mathcal{J}_i \subseteq \{1, 2, \dots, J\}$$

be a subset of cardinality $|\mathcal{J}_i| = J - 1$. Using (8.27) and (8.28) we have

$$J = [G_{\widehat{w}}H : G_{\widehat{w}}] = |H| = [l : k].$$

Then letting τ_1 be a coset representative in $G_{\widehat{w}}H$, and letting σ_1 be a coset representative in $G_{\widehat{w}}$, we get

$$\begin{aligned} \|\lambda\|_1 &= \sum_{i=1}^I \sum_{j=1}^J |\lambda(\tau_i \sigma_j)| = \sum_{j=1}^J |\lambda(\tau_1 \sigma_j)| \\ (9.2) \quad &= |\lambda(\tau_1 \sigma_1)| + \sum_{j=2}^J |\lambda(\tau_1 \sigma_j)| = (J - 1) + (J - 1) \\ &= 2([l : k] - 1). \end{aligned}$$

The following result establishes the existence of relative Minkowski units.

Theorem 9.1. *Let β in F_l be a special Minkowski unit with respect to the infinite place \widehat{w} .*

(i) *If l/\mathbb{Q} is a totally real Galois extension, then the elements in the set*

$$(9.3) \quad \{\tau_i \sigma_j (\Delta(\beta, \lambda)) : i = 1, 2, \dots, I \text{ and } j \in \mathcal{J}_i\}$$

are multiplicatively independent relative units in $E_{l/k}$, and satisfy

$$(9.4) \quad h(\Delta(\beta, \lambda)) \leq 2([l : k] - 1)h(\beta).$$

(ii) If l/\mathbb{Q} is a totally complex Galois extension, then the elements in the set

$$(9.5) \quad \{\tau_i \sigma_j(\Delta(\beta \rho(\beta)), \lambda) : i = 1, 2, \dots, I \text{ and } j \in \mathcal{J}_i\}$$

are multiplicatively independent relative units in $E_{l/k}$, and satisfy

$$(9.6) \quad h(\Delta(\beta \rho(\beta), \lambda)) \leq 4([l : k] - 1)h(\beta).$$

Proof. By Lemma 3.3 the functions

$$\{[\tau_i \sigma_j, \lambda] : i = 1, 2, \dots, I \text{ and } j \in \mathcal{J}_i\} \subseteq \mathcal{L}_{l/k}$$

are linearly independent. Let β in F_l be a special Minkowski unit. Corollary 8.1 implies that the algebraic numbers

$$(9.7) \quad \{\Delta(\beta, [\tau_i \sigma_j, \lambda]) : i = 1, 2, \dots, I \text{ and } j \in \mathcal{J}_i\}$$

are multiplicatively independent relative units in $E_{l/k}$. If l/\mathbb{Q} is a totally complex Galois extension such that

$$(9.8) \quad G_{\hat{w}} = \{1, \rho\}, \quad \text{where } \rho^2 = 1,$$

then it follows from Lemma 5.3 that $\beta \rho(\beta)$ is a special Minkowski unit with respect to \hat{w} . In this case Corollary 8.1 asserts that the algebraic numbers

$$(9.9) \quad \{\Delta(\beta \rho(\beta), [\tau_i \sigma_j, \lambda]) : i = 1, 2, \dots, I \text{ and } j \in \mathcal{J}_i\}$$

are multiplicatively independent relative units in $E_{l/k}$.

Assume that l/\mathbb{Q} is a totally real Galois extension. Then it follows from (8.13) in the statement of Lemma 8.2, that

$$(9.10) \quad \tau_i \sigma_j(\Delta(\beta, \lambda)) = \Delta(\beta, [\tau_i \sigma_j, \lambda])$$

for each $i = 1, 2, \dots, I$ and each $j \in \mathcal{J}_i$. Combining (9.7) and (9.10), we find that the conjugate algebraic numbers in the set

$$\{\tau_i \sigma_j(\Delta(\beta, \lambda)) : i = 1, 2, \dots, I \text{ and } j \in \mathcal{J}_i\}$$

are multiplicatively independent relative units in $E_{l/k}$. That is, $\Delta(\beta, \lambda)$ is a relative Minkowski unit in $E_{l/k}$. Applying the inequality (8.8) and (9.2), we get the bound

$$h(\Delta(\beta, \lambda)) \leq \|\lambda\|_1 h(\beta) = 2([l : k] - 1)h(\beta).$$

This verifies the inequality (9.4).

Now assume that l/\mathbb{Q} is a totally complex Galois extension, and let $G_{\hat{w}}$ be given by (9.8). It follows from (8.15) in the statement of Lemma 8.2, that

$$(9.11) \quad \tau_i \sigma_j(\Delta(\beta \rho(\beta), \lambda)) = \Delta(\beta \rho(\beta), [\tau_i \sigma_j, \lambda])$$

for each $i = 1, 2, \dots, I$ and each $j \in \mathcal{J}_i$. In this case we combine (9.7) and (9.11) and conclude that the conjugate algebraic numbers in the set

$$\{\tau_i \sigma_j(\Delta(\beta \rho(\beta), \lambda)) : i = 1, 2, \dots, I \text{ and } j \in \mathcal{J}_i\}$$

are multiplicatively independent relative units in $E_{l/k}$. Therefore $\Delta(\beta \rho(\beta), \lambda)$ is a relative Minkowski unit in $E_{l/k}$. It follows from the bound (8.8) and (9.2), that

$$\begin{aligned} h(\Delta(\beta \rho(\beta), \lambda)) &\leq \|\lambda\|_1 h(\beta \rho(\beta)) \\ &\leq 2([l : k] - 1)(h(\beta) + h(\rho(\beta))) \\ &= 4([l : k] - 1)h(\beta). \end{aligned}$$

This establishes the inequality (9.6). \square

We now prove Theorem 1.2. Let l/\mathbb{Q} be a Galois extension with N archimedean places, and let \widehat{w} be a particular archimedean place of l . Let $\eta_1, \eta_2, \dots, \eta_{r(l)}$ be a basis for the group F_l , where $r(l) = N - 1$. By Theorem 5.1 there exists a special Minkowski unit β with respect to \widehat{w} , such that

$$(9.12) \quad h(\beta) \leq 2 \sum_{j=1}^{r(l)} h(\eta_j).$$

If l/\mathbb{Q} is totally real, then it follows from (i) in the statement of Theorem 9.1 that

$$\gamma = \Delta(\beta, \lambda)$$

is a relative Minkowski unit for the subgroup $E_{l/k}$. Combining the inequalities (9.4) and (9.12), we find that

$$(9.13) \quad h(\gamma) = h(\Delta(\beta, \lambda)) \leq 2([l : k] - 1)h(\beta) \leq 4([l : k] - 1) \sum_{j=1}^{r(l)} h(\eta_j),$$

and this verifies (1.10).

Now suppose that l/\mathbb{Q} is a totally complex Galois extension. In this case the stabilizer $G_{\widehat{w}}$ is cyclic of order 2, and we use (9.1). It follows from (ii) in the statement of Theorem 9.1 that

$$\gamma = \Delta(\beta\rho(\beta), \lambda)$$

is a relative Minkowski unit for the subgroup $E_{l/k}$. To complete the proof we combine the inequalities (9.6) and (9.12). We find that

$$h(\gamma) = h(\Delta(\beta\rho(\beta), \lambda)) \leq 4([l : k] - 1)h(\beta) \leq 8([l : k] - 1) \sum_{j=1}^{r(l)} h(\eta_j),$$

and this proves (1.11).

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